Combining Price Indices in Temporal Hierarchies*

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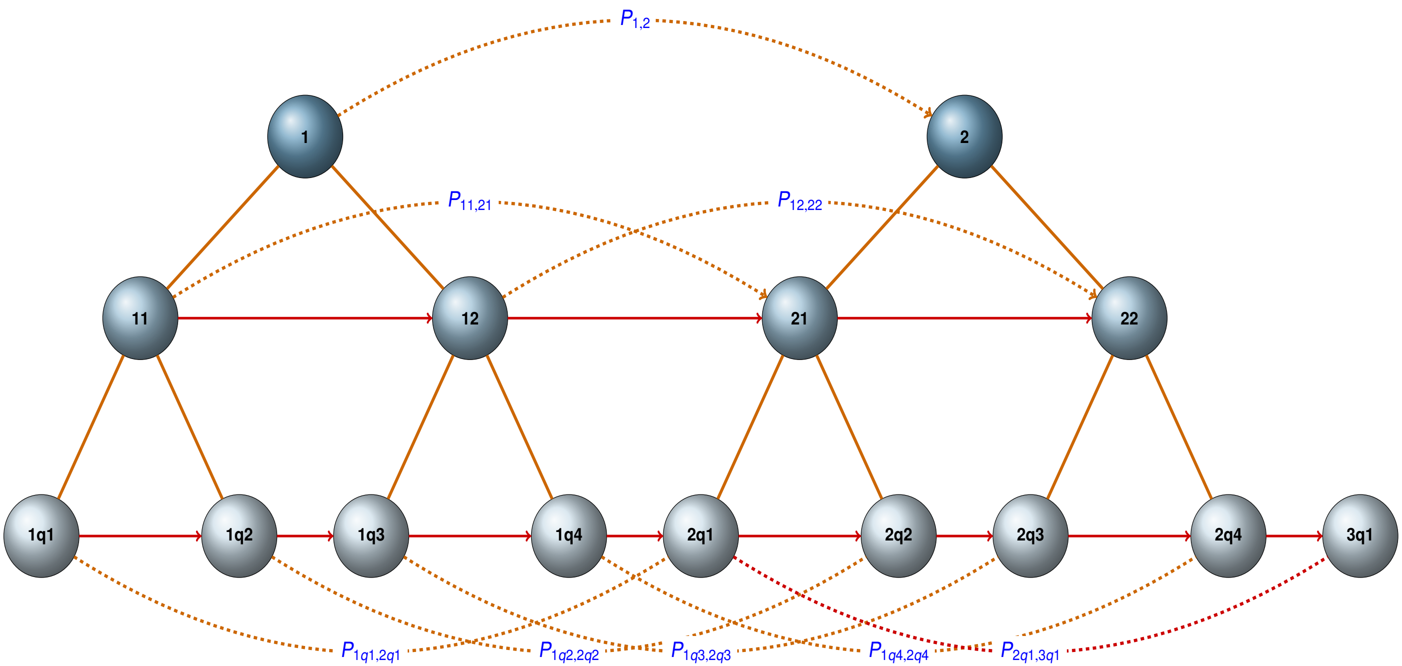
Measuring Asset Price Changes

- Since the global financial crisis, central banks have become more aware of how developments in the housing market can affect the rest of the economy, and in some cases **threaten financial stability**.
- Lower frequency indices are fairly **robust** to the estimation method and model specification (Diewert et al, 2011; Hill and Scholz, 2018; Hill et al., 2020a).
- Higher frequency indices (monthly and higher) can yield **significantly different conclusions** depending on the method and model specification (Hill et al., 2020a).
- Higher and lower frequency indices can show **different trends**, and hence be **inconsistent** with each other.

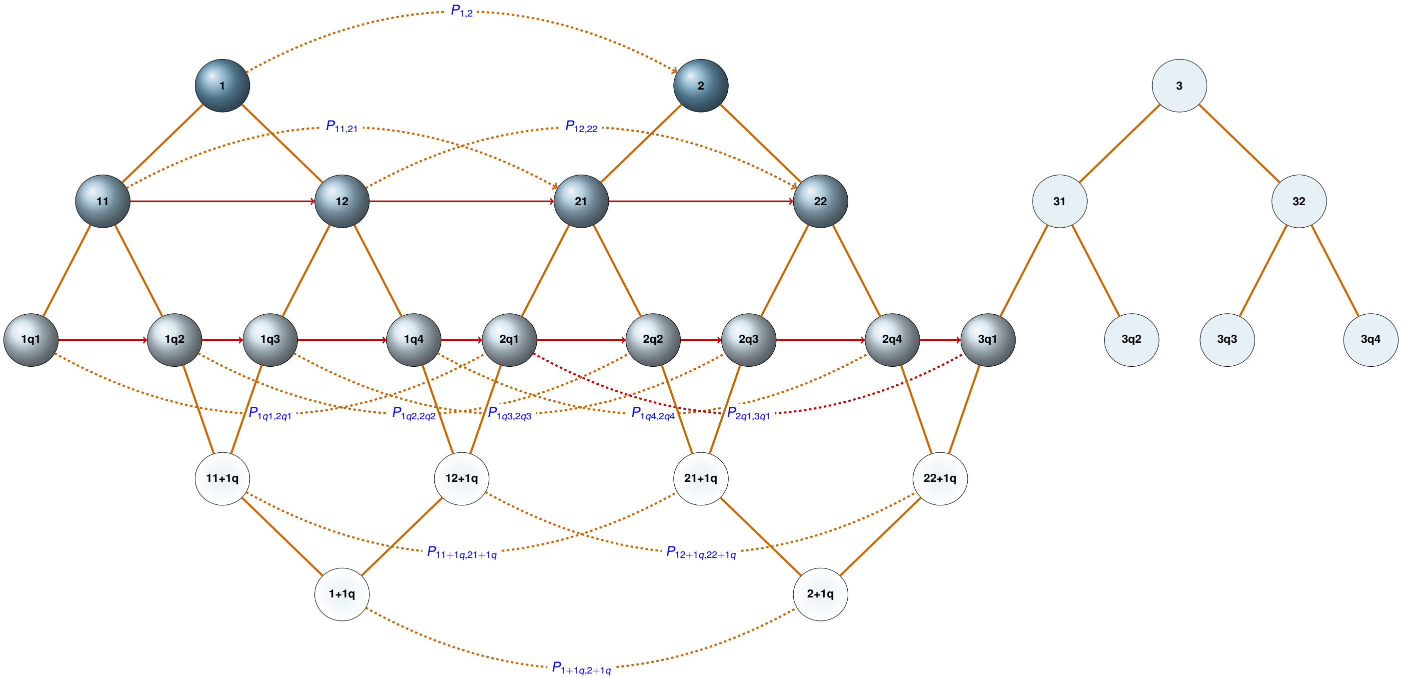
Our Work

- We arrange price indices in **temporal hierarchies**:
  - The basic building block is the time period over which the highest frequency index is defined (e. g. monthly).
  - The second highest frequency consists of a whole number of highest frequency periods (e. g., three months).
  - The next frequency consists of a whole number of periods from the previous layer in the hierarchy (e.g., four quarters), etc.
- In addition to producing **improved indices at all frequencies**, these indices are also produced in **real time**. Our method produces a new annual and quarterly index every month.

Annual, Semi-Annual,Quarterly



Adding RealTime



Relationship to Other Works

- Our method is related to:
  - The least-squares reconciliation approach for temporal hierarchies of Athanasopoulos et al. (2017), which in turn draws on Hyndman et al. (2011), and Hyndman et al. (2016).
  - The multilateral price index literature, and especially the Gini-Eltetö-Szulc (GEKS) method (see, for example, Diewert, 1999, and Balk, 2008).
- Athanasopoulos et al. (2017) focuses on series that can be summed across time periods.
- We formulate different combinations of indices that provide alternative answers to the same question (similar to GEKS). Our identifying restrictions require reconciled indices asking the same question to give the same answer.

Our Contribution

- Propose a simple method to construct **reconciled annualised** (year-on-year) price indices across temporal hierarchies (e.g. annual, quarterly, monthly) and recover the **reconciled period-on-period** higher frequencies (e.g. quarterly and monthly)
- Propose a method that can be used if **transaction level data** are available, as well as with **commercially available price indices**.
- Show the link between two literatures (reconciling forecasts and multilateral price indices construction)

The Simplest Case–Two layers

- Notation:
  - $P_{1,2}$ the price change from year 1 and 2.
  - $P_{11,12}$ the price change from the 1st half of year 1 to the 1st half of year 2,
  - $P_{21,22}$ the price change from the 2nd half of year 1 to the 2nd half of year 2.
- **Note:** The geometric mean of $P_{11,21}$ and $P_{12,22}$, is an alternative measure to $P_{1,2}$
- **Objective:** Alter the original indices $P_{1,2}$, $P_{11,21}$ and $P_{12,22}$ by the logarithmic-least-squares amount necessary to reconcile our two annualized indices.
- **Reconciliation means:**

$$\ln \hat{P}_{1,2} = 0.5(\ln \hat{P}_{11,21} + \ln \hat{P}_{12,22})$$

The Least-Squares Problem

$$\text{Min}_{\ln \hat{P}_{1,2}, \ln \hat{P}_{11,21}, \ln \hat{P}_{12,22}} \left[(\ln \hat{P}_{1,2} - \ln P_{1,2})^2 + 0.5(\ln \hat{P}_{11,21} + \ln \hat{P}_{12,22} - \ln P_{11,21} - \ln P_{12,22})^2 \right],$$

$$\text{such that} \quad \ln \hat{P}_{1,2} = 0.5(\ln \hat{P}_{11,21} + \ln \hat{P}_{12,22}). \qquad (1)$$

- We can rewrite this problem more compactly in **matrix notation** as follows:

$$y = S\beta + \varepsilon \qquad (2)$$

where

$$S = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad y = \begin{pmatrix} \ln P_{1,2} \\ 0.5(\ln P_{11,21}) \\ 0.5(\ln P_{12,22}), \end{pmatrix}$$

and ε is an error vector representing the aggregation error with zero mean and covariance matrix Σ.

- Hyndman et al. (2011, 2016) proposed a variant on this linear model in the context of reconciliation of forecasts.
- They showed that when the aggregation errors approximately satisfy the same aggregation structure as the original data, then OLS and GLS estimates of β are identical.
- Even if the aggregation errors do not satisfy this assumption, they argue the OLS solution will still be a consistent way of reconciling the base forecast.

Three Layer Case

- Now we have three reconciliation equations:

  (i) $\hat{P}_{1,2} = (\hat{P}_{1q1,2q1} \times \hat{P}_{1q2,2q2} \times \hat{P}_{1q3,2q3} \times \hat{P}_{1q4,2q4})^{1/4}$

  (ii) $(\hat{P}_{11,21}) = (\hat{P}_{1q1,2q1} \times \hat{P}_{1q2,2q2})^{1/2}$

  (iii) $(\hat{P}_{12,22}) = (\hat{P}_{1q3,2q3} \times \hat{P}_{1q4,2q4})^{1/2}$

- Three more equations relating the reconciled prices indices can be derived from (i), (ii) and (iii). These are the following:

  (iv) $\hat{P}_{1,2} = (\hat{P}_{11,21} \times \hat{P}_{12,22})^{1/2}$.

  (v) $\hat{P}_{1,2} = [(\hat{P}_{1q1,2q1} \times \hat{P}_{1q2,2q2})^{1/2} \times \hat{P}_{12,22}]^{1/2}$.

  (vi) $\hat{P}_{1,2} = [\hat{P}_{11,21} \times (\hat{P}_{1q3,2q3} \times \hat{P}_{1q4,2q4})^{1/2}]^{1/2}$.

- **Objective:** (i), (ii), and (iii) are satisfied.

$$y = S\beta + \varepsilon,$$

where

$$S = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad y = \begin{pmatrix} \ln P_{1,2} \\ 0.5(\ln P_{11,21}) \\ 0.5(\ln P_{12,22}) \\ 0.25(\ln P_{1q1,2q1}) \\ 0.25(\ln P_{1q2,2q2}) \\ 0.25(\ln P_{1q3,2q3}) \\ 0.25(\ln P_{1q4,2q4}) \end{pmatrix}, \qquad (3)$$

and ε again denotes an error vector.

- **Solution:**

$$\hat{y} = S\hat{\beta} = S(S'S)^{-1}S'y \tag{4}$$

$$\begin{aligned} \ln \hat{P}_{1,2} &= \frac{1}{21}\{\mathbf{12}\ln P_{1,2} + \mathbf{6}[\frac{1}{2}(\ln P_{11,21} + \ln P_{12,22})] \\ &+ \mathbf{3}[\frac{1}{4}(\ln P_{1q1,2q1} + \ln P_{1q2,2q2} + \ln P_{1q3,2q3} + \ln P_{1q4,2q4})]\} \end{aligned} \tag{5}$$

$$\begin{aligned} \ln \hat{P}_{11,21} &= \frac{1}{21}\{\mathbf{10}\ln P_{11,21} + \mathbf{5}[\frac{1}{2}(\ln P_{1q1,2q1} + \ln P_{1q2,2q2})] \\ &+ \mathbf{4}(2\ln P_{1,2} - \ln P_{12,22}) \\ &+ \mathbf{2}[2\ln P_{1,2} - \frac{1}{2}(\ln P_{1q3,2q3} + \ln P_{1q4,2q4})]\}. \end{aligned} \tag{6}$$

- This solutions can be reinterpreted as **weighted geometric means** of competing unreconciled indices answering the same question.

Weighted Reconciliation

- Hyndman et al (2016) discuss the optimally reconciled forecasts as those given by the generalised least squares (GLS) solution,

$$\hat{y} = S\tilde{\beta} = S(S'\Sigma^{\dagger}S)^{-1}S'\Sigma^{\dagger}y, \quad (7)$$

  where, $\Sigma^{\dagger}$ is the generalised inverse of the covariance matrix of ε in the model in (2). However, $\Sigma^{\dagger}$ is unknown and virtually impossible to estimate.

- An alternative might be to use weighted least squares (WLS). That is, replacing $\Sigma^{\dagger}$ by W, a diagonal matrix with elements equal to the inverse of the variances of the elements of ε,

$$\hat{y}^{WLS} = S\tilde{\beta} = S(S'WS)^{-1}S'Wy, \quad (8)$$

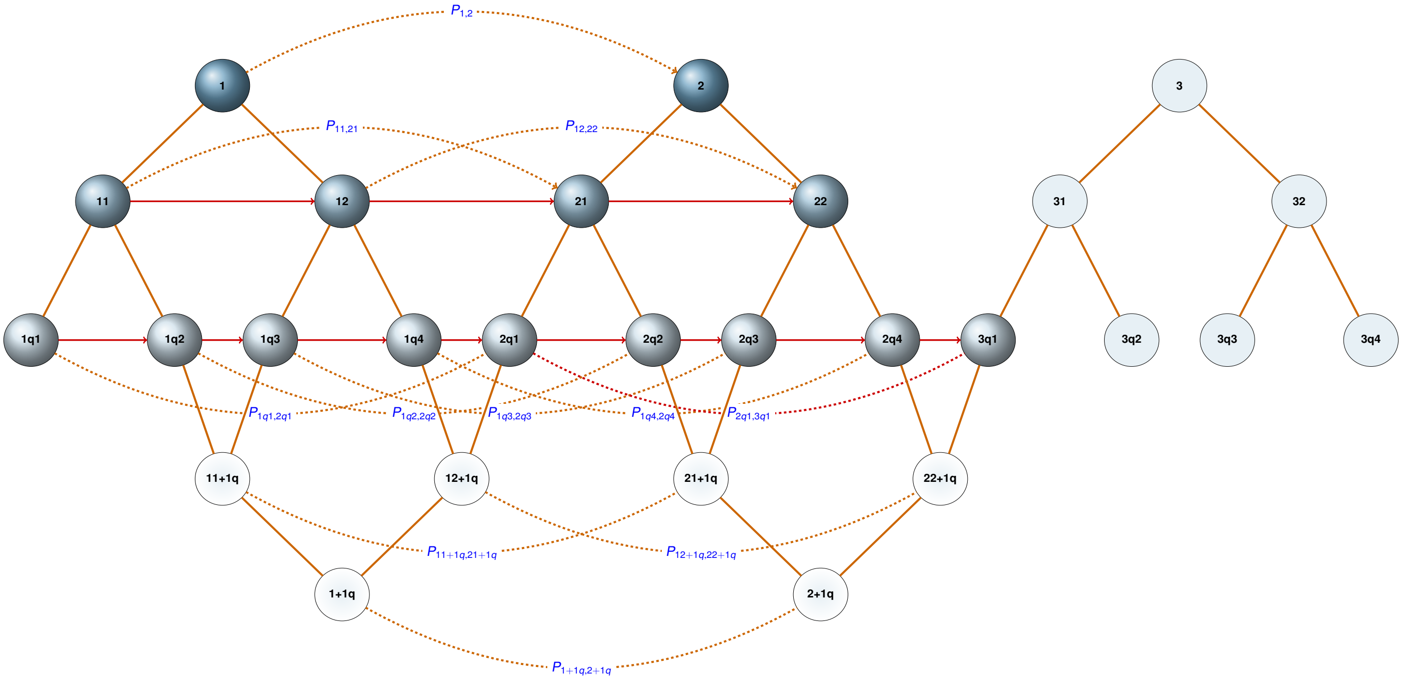
Time Series Dimension of the Reconciliation

- So far y defined as a stacked set of annualised indices at a point in time.
- For real time, use a **time-varying parameter model**, maintaining the structure and assumptions of the reconciliation

$$y_t = S\beta_t + \epsilon_t; \qquad t = 1, \ldots, T \tag{9}$$

- Under **WLS** assumption: $\epsilon_t \sim N(0, H_t)$, where $H_t = (W_{TW_t})^{-1}$; **Spherical** assumption: $H_t = \sigma_\varepsilon^2 I$
- Assume $\beta_t = \beta_{t-1} + \eta_t$ and the covariance of η_t is $Q = diag((S'S)^{-1})$ which can be easily verified to be $Q = \sigma_\eta^2 I$, where σ_η^2 is a constant and I is an identity matrix.
- Assume at $t = 0$, the covariance of β_t is Q
- Estimate with a **Kalman Filter**

Backing out reconciled higher frequencies



Backing Out Higher Frequency Indices

- **Recursive** algorithms (e.g. 2nd quarter)

$$p^R_{1q1,2q1} + p^R_{2q1,2q2} = p^R_{1q1,1q2} + p^R_{1q2,2q2}$$

This can be rearranged as follows:

$$p^R_{2q1,2q2} = p^R_{1q1,1q2} + p^R_{1q2,2q2} - p^R_{1q1,2q1}. \tag{10}$$

- A **system** of equations approach ($t = 1, ..., T$)

$$\begin{aligned} p^R_{1q1,1q2} + p^R_{1q2,1q3} + p^R_{1q3,1q4} + p^R_{1q4,2q1} &= p^R_{1q1,2q1} \\ p^R_{1q2,1q3} + p^R_{1q3,1q4} + p^R_{1q4,2q1} + p^R_{2q1,2q2} &= p^R_{1q2,2q2} \\ &\vdots \\ p^R_{(T-1)q4,Tq1} + p^R_{Tq1,Tq2} + p^R_{Tq2,Tq3} + p^R_{Tq3,Tq4} &= p^R_{(T-1)q4,Tq4} \end{aligned} \tag{11}$$

Backing Out Higher Frequency Indices: Initial Conditions

- The reconciliation produces 7 reconciled year-on-year (y-o-y), there are 10 period-on-period (p-o-p) links.
- We need an initial condition in the algorithms:
  - For the recursive, set **equal to the unreconciled quarterly indices for the first year**:

$$p^R_{1q1,1q2} = p_{1q1,1q2}, \quad p^R_{1q2,1q3} = p_{1q2,1q3}, \textit{etc.} \tag{12}$$

  - For the system all possible combinations (of consecutive indices blocks) and average

Backing Out Higher Frequency Indices: Practical Issues

- A bit of algebra shows the recursive formulation induces a **spurious memory** of lagged terms. Two possible alternatives:
  (i) **no memory** $p^R_{2q1,2q2} = p_{1q1,1q2} + p^R_{1q2,2q2} - p^R_{1q1,2q1}$
  (ii) **average** $p^R_{2q1,2q2} = [(p^R_{1q1,1q2} + p_{1q1,1q2})/2] + p^R_{1q2,2q2} - p^R_{1q1,2q1}$

- The system of equations approach **leads to a revision of the whole history**. Two possible alternatives:
  (i) **rolling window (RW)**: Fixed the number of quarters (months) in the rolling window
  (ii) **combination - system recursive**: System for n_q and recursive (average) after

Measuring the Quality of an Index - Which do we choose?

- Use a variant on the quality measure proposed by Hill et al. (2020a) based on **repeat-sales**.
- Suppose a property i sells in periods t and $t+k$. For this repeat sale we can compare the **actual observed price change** $p_{i,t+k}/p_{i,t}$ with the **corresponding price change obtained from an index,** P_{t+k}/P_t

$$d_i = \ln\left(\frac{P_{t+k}}{P_t}\right) - \ln\left(\frac{p_{i,t+k}}{p_{i,t}}\right). \tag{13}$$

- Averaging over all repeat-sales properties i, N_{RS} in our records, our measure of index quality is given by :

$$IQ = \frac{1}{N_{RS}}\sum_{i=1}^{N_{RS}}(d_i)^2 \tag{14}$$

- The index with the smallest IQ is preferred.

- Application to Sydney Eastern Suburbs with Real Time Reconciliation, Period: 2001–2014
- Three-level hierarchy (Annual, Quarterly, Monthly)

Suburb	Postcode	total transactions	repeat-sales
Paddington	2021	2535	486
Bondi Junction	2022	1499	281
Bellevue Hill	2023	1018	153
Waverley	2024	1241	203
Woollahra	2025	1203	224
Bondi	2026	2287	393
Edgecliff	2027	350	51
Double Bay	2028	350	61
Rose Bay	2029	760	117
Vaucluse	2030	1963	270
Randwick	2031	2527	432
Kingsford	2032	1044	151
Kensington	2033	627	78
Coogee	2034	1301	192
Pagewood	2035	2784	442
Matraville	2036	1965	263

Reconciled Annual

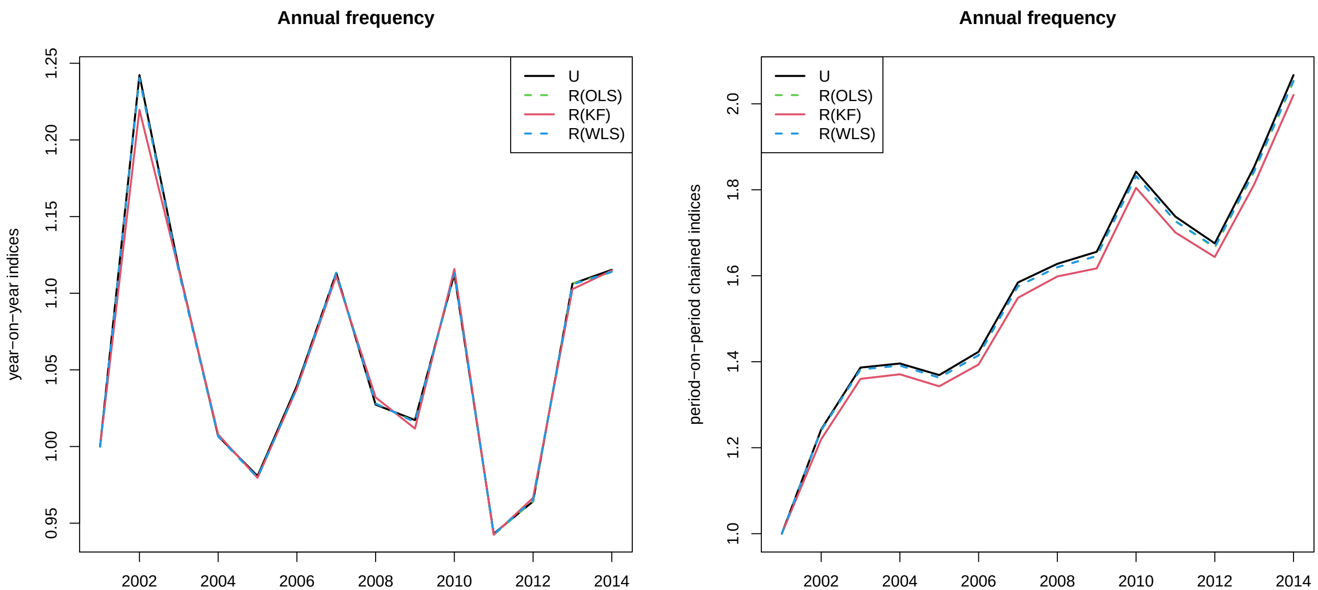


Figure: Left: year-on-year indices, right: chained period-on-period indices. U unreconciled index, R(OLS) reconciled using OLS prediction, R(KF) reconciled index using Kalman Filter prediction, and R(WLS) reconciled using Weighted Least Squares prediction

Reconciled Quarterly

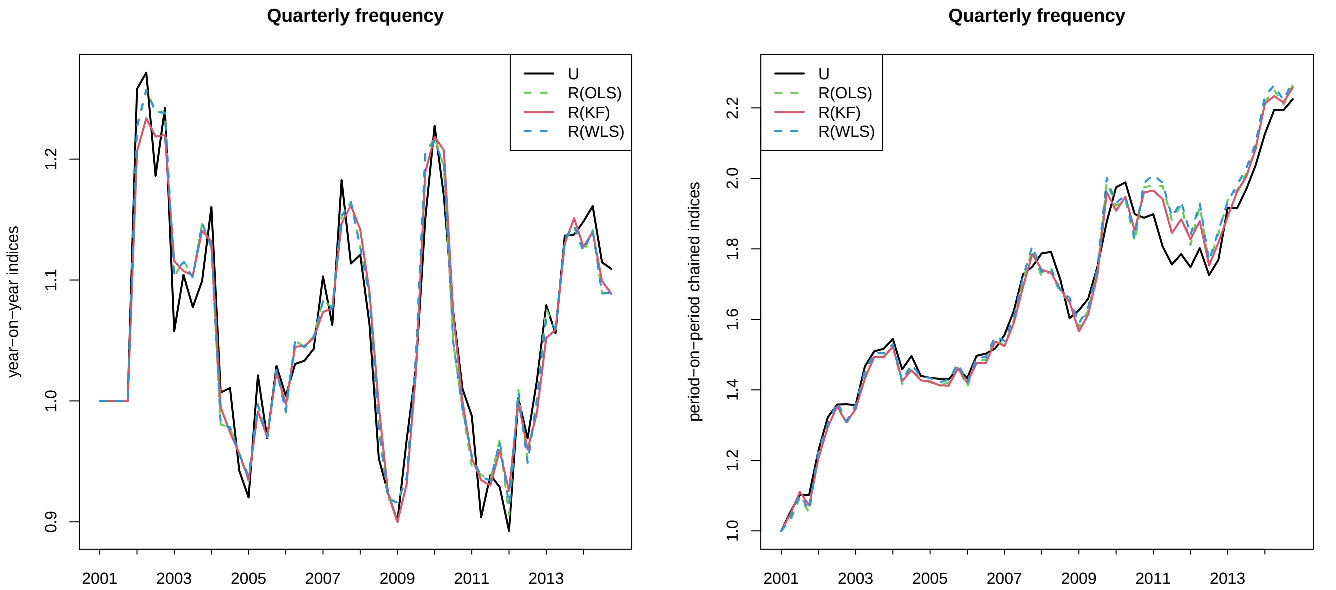


Figure: Left: year-on-year indices, right: chained period-on-period indices (system recursive). U unreconciled index, R(OLS) reconciled using OLS prediction, R(KF) reconciled index using Kalman Filter prediction, and R(WLS) reconciled using Weighted Least Squares prediction

Reconciled Monthly

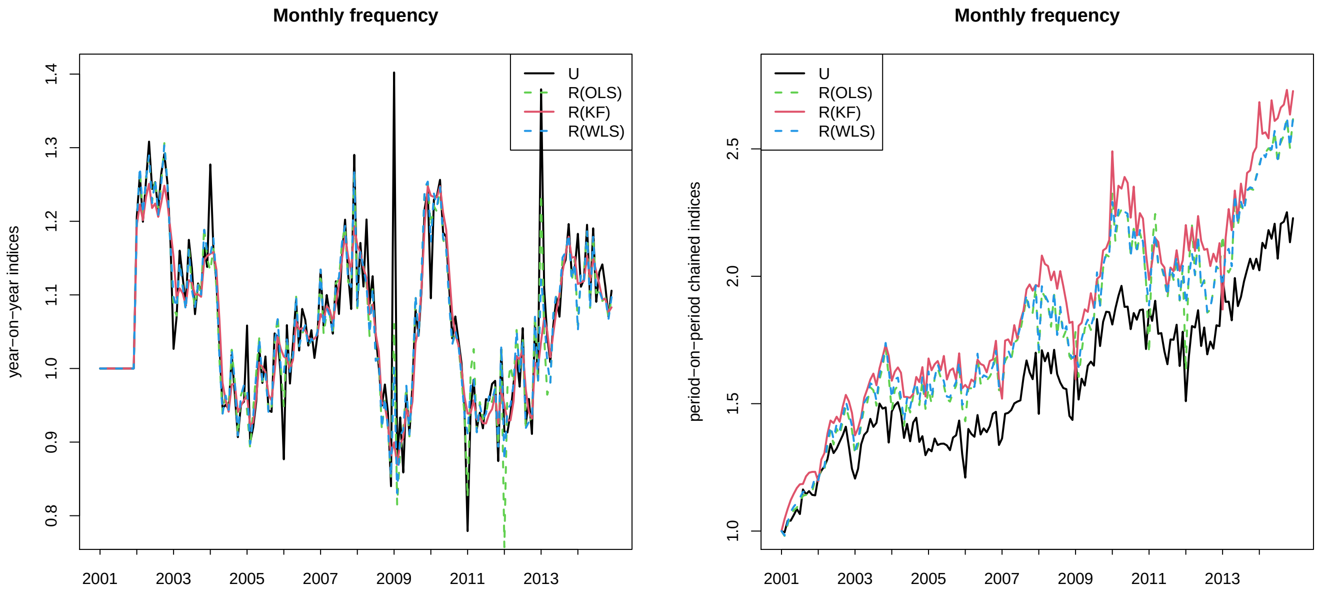


Figure: Left: year-on-year indices, right: chained period-on-period indices.(system recursive). U unreconciled index, R(OLS) reconciled using OLS prediction, R(KF) reconciled index using Kalman Filter prediction, and R(WLS) reconciled using Weighted Least Squares prediction

Table: Measuring the quality of the reconciled indices

Frequency	Method	U	R(OLS)	R(KF)	R(WLS)
Annual		0.042507			
			0.042601	**0.042237**	0.042556
Quarterly		0.041189			
	pure recursive		0.104763	0.101598	0.104730
	no memory		0.041599	**0.041077**	0.041582
	averaged		0.041675	**0.040693**	0.041682
	full system		0.042520	0.041801	0.042374
	system recursive		**0.040571** (40)	**0.040339** (40)	**0.040541** (40)
	RW system		0.041277 (15)	**0.041095** (16)	**0.041182** (15)
Monthly		0.043518			
	pure recursive		0.111889	0.108850	0.111946
	no memory		0.045119	0.044120	0.044887
	averaged		0.045262	**0.043217**	0.044803
	full system		**0.043378**	**0.041794**	**0.042530**
	system recursive		**0.040727** (46)	**0.040723** (47)	**0.040516** (46)
	RW system		**0.043271** (62)	**0.041588** (49)	**0.042349** (61)

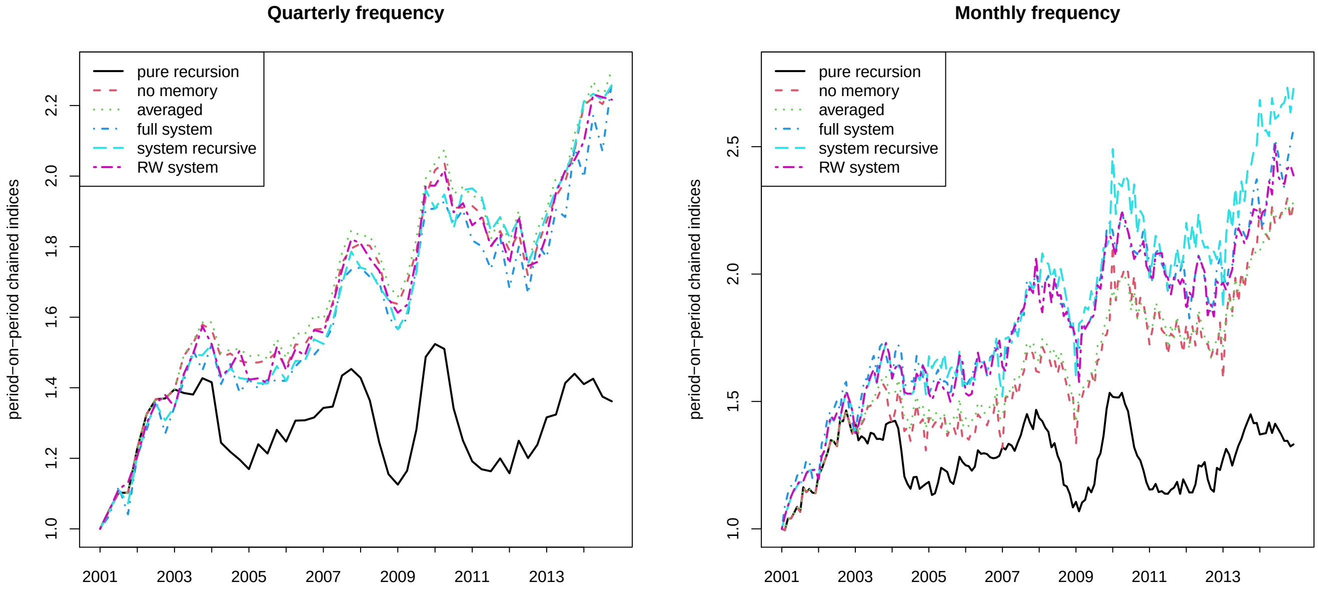


Figure: Backed-Out Indices using Kalman Filter Predictor Reconciliation

Reconciling Off-The-Shelf Indices

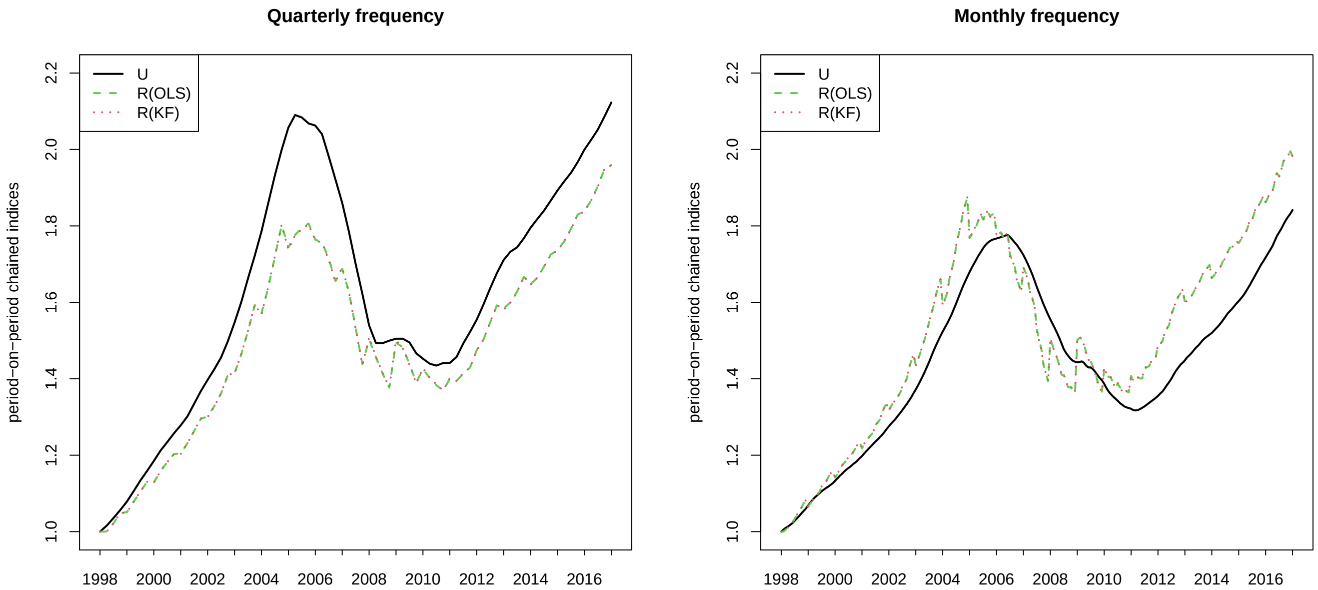


Figure: Off the shelf US Indices - Residential (FHFA, CS, ZIHV (SAdj)). Using: R(KF) and full system p-o-p recovery

Summary and Outlook (cont.)

- We arrange price indices in temporal hierarchies.
- The basic building block is the time period over which the highest frequency index is defined (e.g. monthly).
- Propose a simple method to construct reconciled annualised (year-on-year) price indices across temporal hierarchies (e.g. annual, quarterly, monthly) and recover the reconciled period-on-period higher frequencies (e.g. quarterly and monthly)
- Propose a method that can be used if transaction level data are available, as well as with commercially available price indices.
- Show the link between two literatures (reconciling forecasts and multilateral price indices construction)

Thank you for your attention!

Literature

- Athanasopoulos, G., R. J. Hyndman, N. Kourentzes, and F. Petropoulos (2017), “Forecasting with temporal hierarchies,” *European Journal of Operational Research* 262(1), 60-74.
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