Gini vs. Dalton:
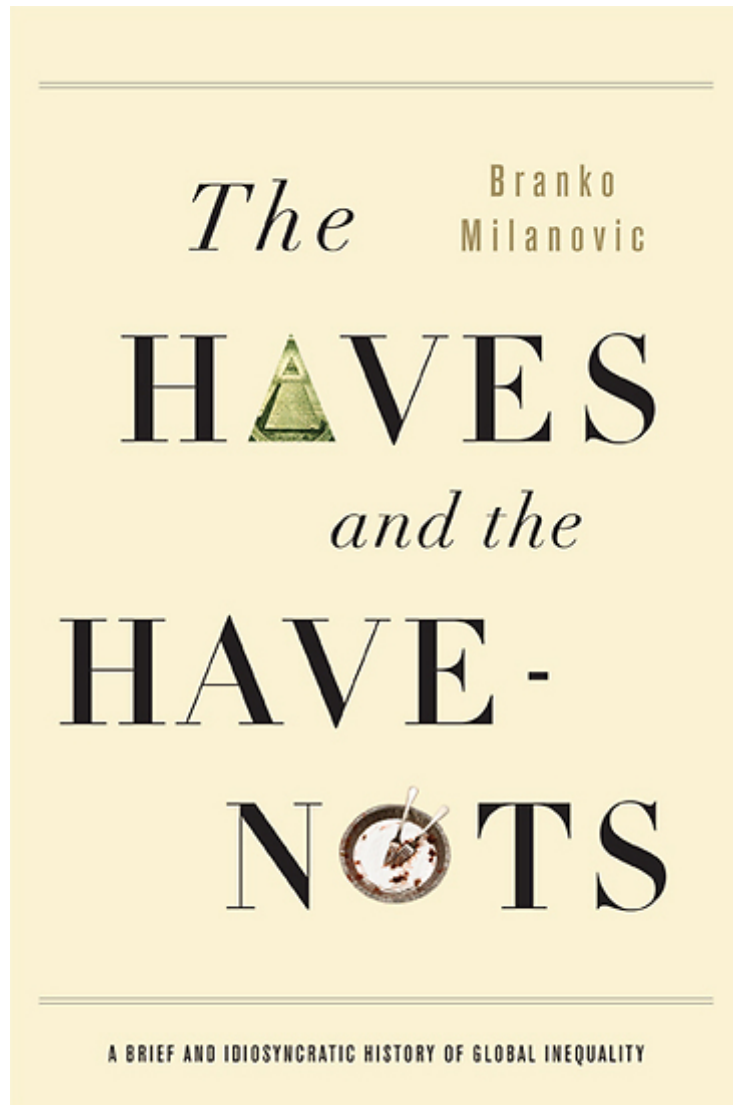# Mathematical Tools and Value Judgements in the Measurement of Income Inequality

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The HAVES and the HAVE-NOTS

Branko Milanovic

A BRIEF AND IDIOSYNCRATIC HISTORY OF GLOBAL INEQUALITY

Branko Milanovic

'Gini by far the most popular index of inequality'

End of the story?

ANASTASIA

At the end of three days, moving southward, you come upon Anastasia, a city with concentric canals watering it and kites flying over it. Such is the power, sometimes called malignant, sometimes benign, that Anastasia, the treacherous city, possesses; if for eight hours a day you work as a cutter of agate, onyx, chrysoprase, your labor which gives form to desire takes from desire its form; and you believe you are enjoying Anastasia wholly when you are only its slave.

#Invisible Cities #Italo Calvino #Anastasia #artists on tumblr #joe kuth 1 May 2014

#Invisible Cities #Italo Calvino #Zobeide #artists on tumblr #Matt Kish 16 July 2014
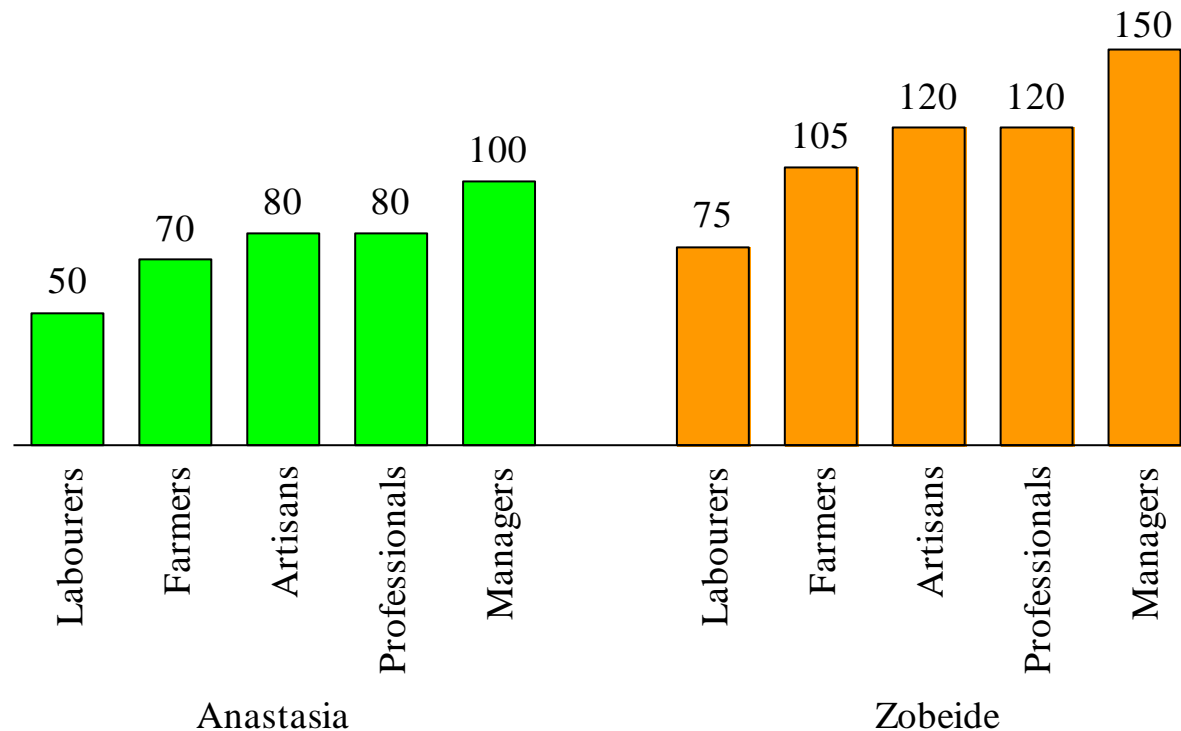
ZOBEIDE

From there, after six days and seven nights, you arrive at Zobeide, the white city, well exposed to the moon, with streets wound about themselves as in a skein. They tell this tale of its foundation: men of various nations had an identical dream. They saw a woman running at night through an unknown city; she was seen from behind, with long hair, and she was naked. They dreamed of pursuing her. As they twisted and turned, each of them lost her. After the dream they set out in search of that city; they never found it, but they found one another; they decided to build a city like the one in the dream.
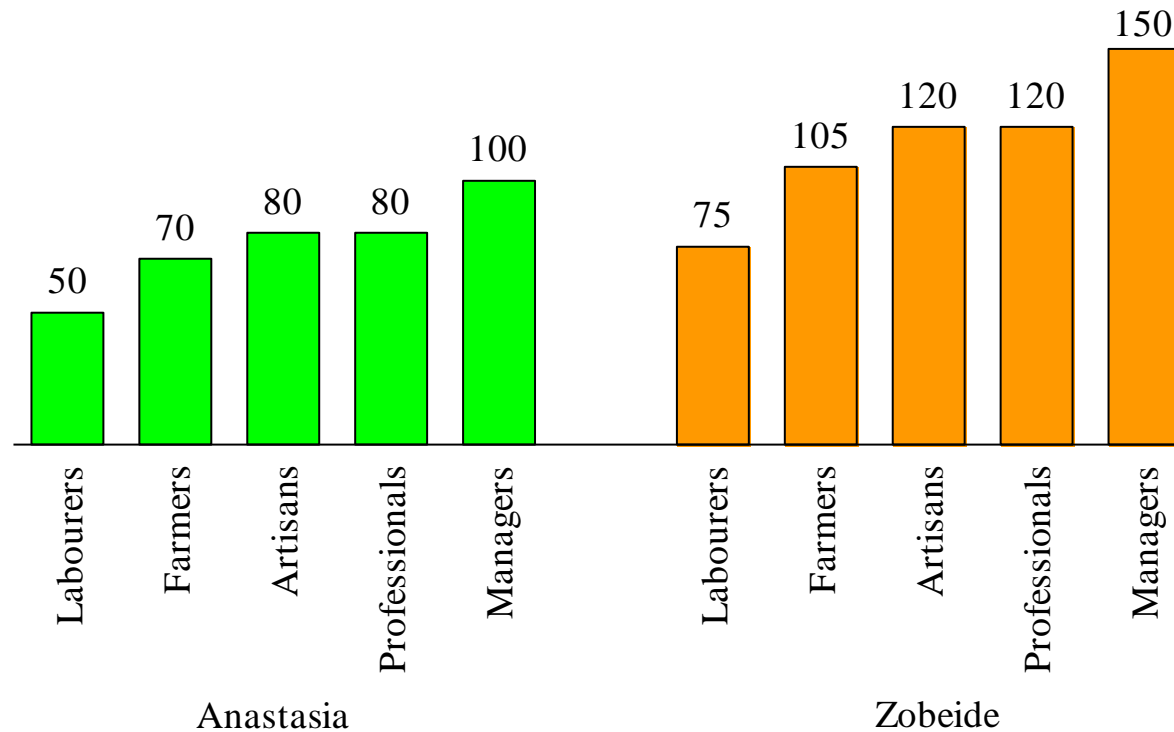
The Traveller's problem

The inhabitants of **Anastasia** and **Zobeide** are divided in 5 equally-sized classes differing for income level, but otherwise identical

Anastasia: Labourers 50, Farmers 70, Artisans 80, Professionals 80, Managers 100

Zobeide: Labourers 75, Farmers 105, Artisans 120, Professionals 120, Managers 150

Which distribution is **more unequal**?

The Accountant's answer

Inequality of incomes is exactly the same in Anastasia and Zobeide

Anastasia: Labourers 50, Farmers 70, Artisans 80, Professionals 80, Managers 100

Zobeide: Labourers 75, Farmers 105, Artisans 120, Professionals 120, Managers 150

Same proportions across all occupations:
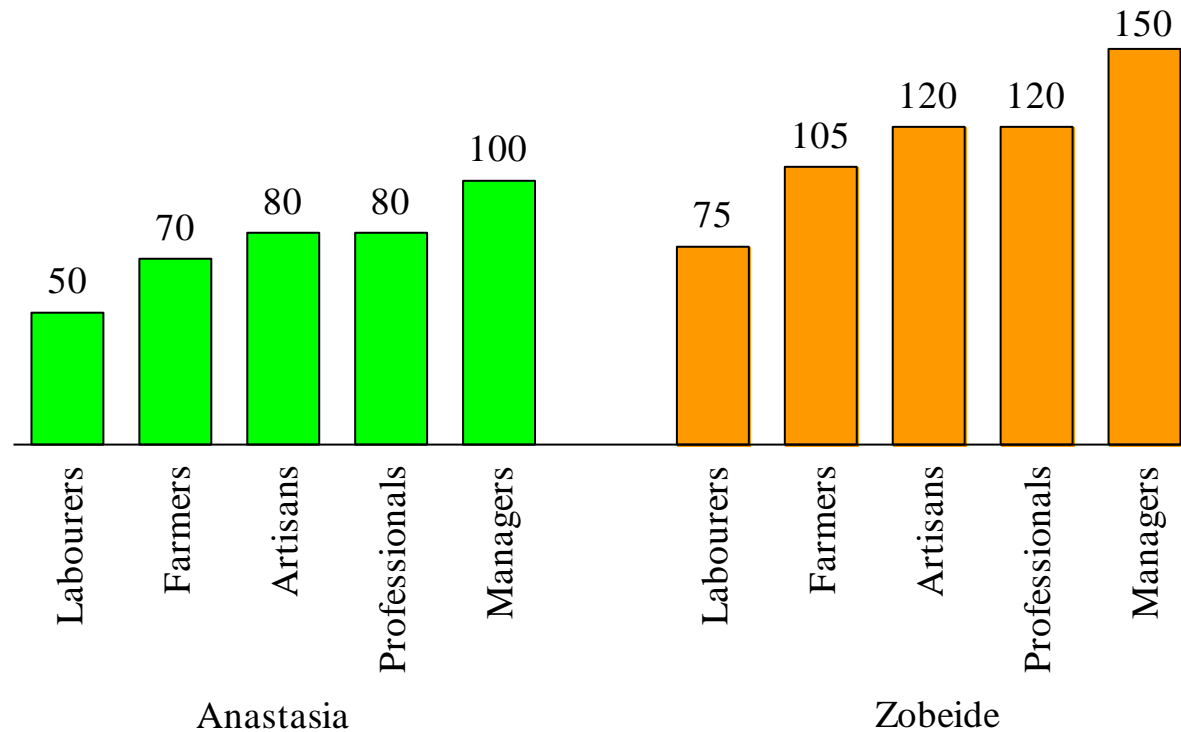*Zobeide incomes = **1.5** × Anastasia incomes*

The Physician's answer

Incomes are more unequally distributed in Zobeide than Anastasia
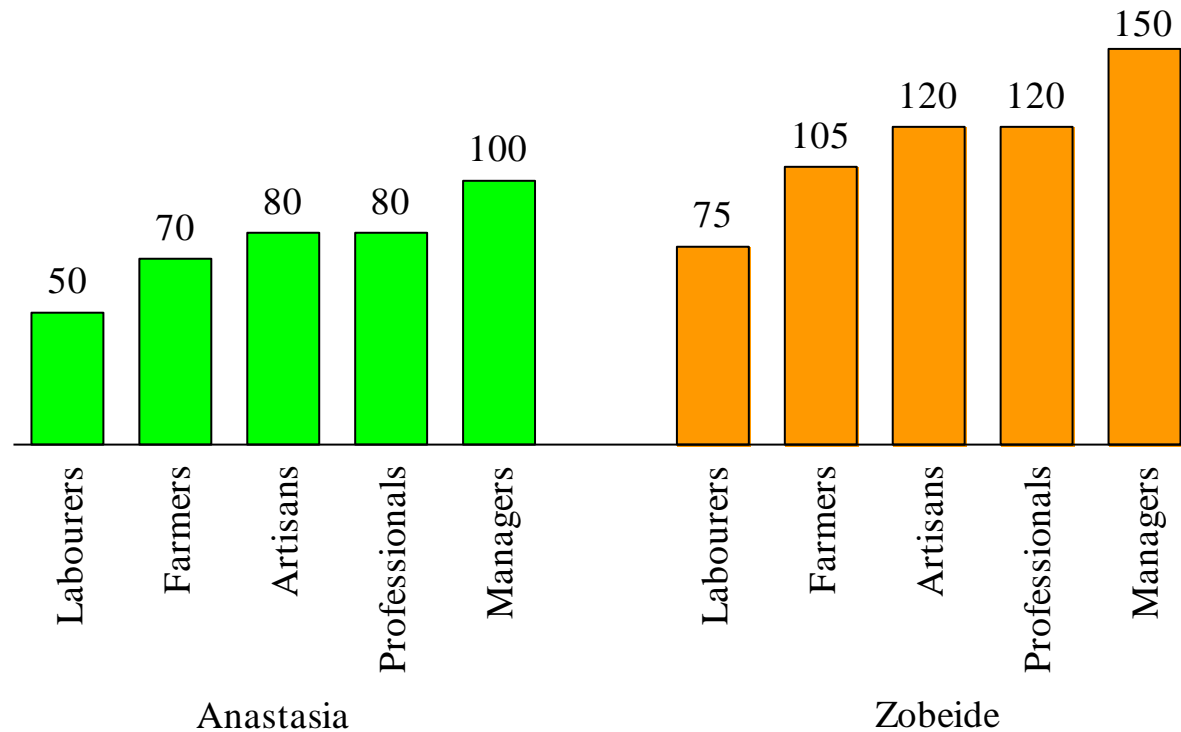
Managers earn 75 ducats more than labourers in Zobeide
vis-à-vis 50 ducats in Anastasia

The Shoemaker's answer

Zobeide is richer and hence preferable to Anastasia

Anastasia — Labourers: 50, Farmers: 70, Artisans: 80, Professionals: 80, Managers: 100

Zobeide — Labourers: 75, Farmers: 105, Artisans: 120, Professionals: 120, Managers: 150

Even if somebody happens to be a labourer, earnings in Zobeide are one half higher than in Anastasia

Measuring inequality historical notes: origins

An old Italian tradition

- Analysis of distribution of income and wealth owes much to Italian economists and statisticians

  > Hugh Dalton in 1920
  > *'the problem of the measurement of the inequality of incomes has not been much considered by English economists. It has attracted rather more attention in America, but it is in Italy that it has hitherto been most fully discussed'*

- Italian lead was not going to last long

Pareto's law (1)

- Pareto (1895, 1897)

- observes that income distribution can be approximated by simple linear relationship in logs:

  $n_x = kx^{-\alpha}$ n_x taxpayers with y≥x, k and α parameters

- Pareto finds that **α**, which he interprets as a ***measure of inequality***, is remarkably constant over time and space

  'the inequality in the distribution of incomes seems therefore to depend much more on the human nature itself than on the economic organisation of the society. It could well be the case that deep modifications of this organisation had but than little impact on the law that governs the distribution of incomes'

- 'Scientific' argument against socialists and radicals

  ⇒ ***Since the outset, empirical analysis of income distribution intertwined with normative assessment***

Pareto's law (2)

- Vigorous international debate

- Bresciani-Turroni (1905)

  '... it is the commonly shared private economic structure of our society, rather than the human nature, to determine the typical shape of the revenue curve, and ... the variations in its shape stem from the special economic conditions which characterise every single economy'

- Bresciani-Turroni's survey in Econometrica (1939) virtually brought to a close this long debate

  □ Pareto's law lacks theoretical foundation and empirical support
  □ Pareto conceived for the first time the idea of measuring inequality, but other indices more appropriate than α, which 'assumes a definite law of distribution of income which sometimes is not in accordance with actual statistics'

Measuring inequality historical notes: turning point no. 1

From α to positive inequality indices (1)

- Pareto adopted a peculiar definition of inequality:

  'in general, when the number of people with income below x falls relative to the number of people with income above x, we shall assert that income inequality declines'

- Many soon criticised Pareto: for most, **α** is an ***index of equality*** (but see Chipman, 1974)

- Search for an alternative inequality index just started

- One limitation of α: accounts for number of taxpayers falling in each income bracket, but ignores amount of incomes they receive

  - Gini (1909, 1910) introduces notion of concentration:

    $\delta = \ln(n_x/n)/\ln(\mu_x n_x/\mu n)$ μ_x mean income of n_x individuals

- Problem with α and δ: meaningful if distribution is Paretian

From α to positive inequality indices (2)

- Gini's (1914) **coefficient of concentration**:

  '*... a measure of concentration which is **independent of the distribution curve** of a variable and allows for the comparison of concentration across the most different variables*'

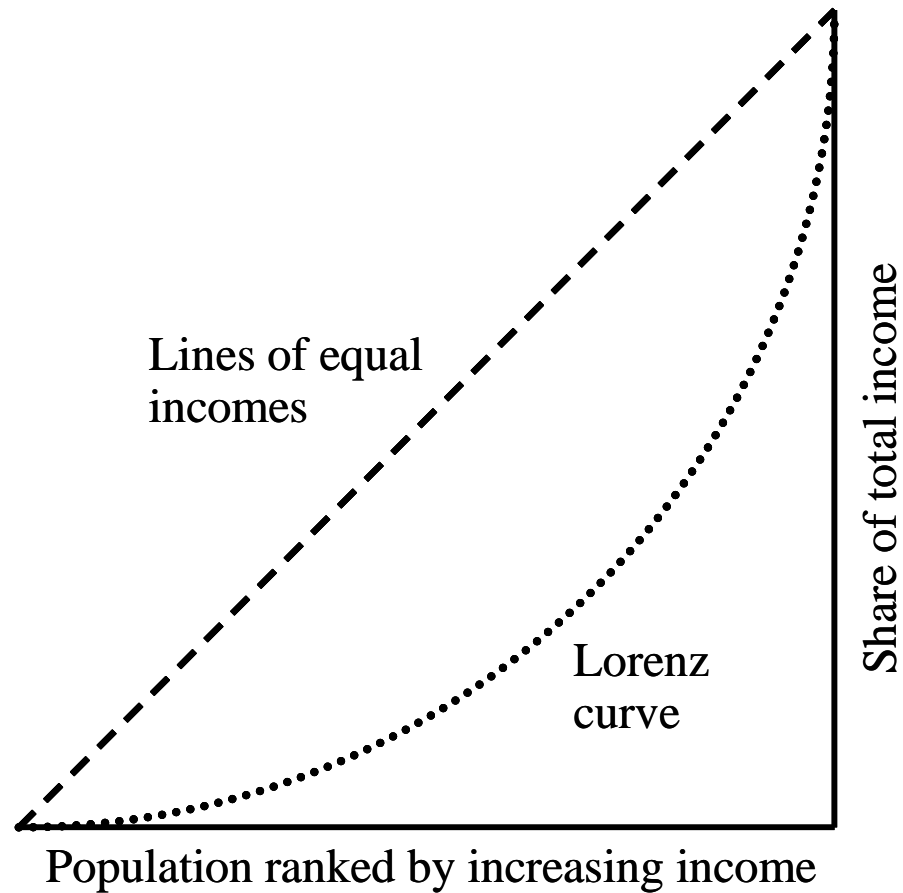 - Half average absolute difference between any possible pair of incomes relative to the mean:

  $$G = \Sigma_i \Sigma_j |y_i - y_j| / 2n^2 \mu$$

  - If we rank incomes from lowest to highest $[y_1 \leq y_2 \leq \ldots \leq y_n]$:

  $$G = \Sigma_i (2i - 1 - n) y_i / n^2 \mu$$

  - Geometrically: twice the area between the 45 degree line and the Lorenz (1905) curve

From α to positive inequality indices (3)

From α to positive inequality indices (4)

- Other positive indices:
  - Relative mean deviation, later labelled maximum equalisation percentage by UN Economic Commission for Europe (1957):

  $$R=\Sigma_i|y_i-\mu|/2n\mu$$

  - Bonferroni (1930) index:

  $$B=\Sigma_i(\mu-\mu_i)/n\mu \quad \text{with } \mu_i=\Sigma_{j\leq i}y_j/i$$

- For positive incomes, all three indices G, R and B are comprised between 0 and 1, with B≥G

- Theil's (1967) indices based on information theory
  - Mean logarithmic deviation:

  $$L= -\Sigma_i\log(y_i/\mu)/n$$

  - Theil index:

  $$T=\Sigma_i(y_i/\mu)\log(y_i/\mu)/n$$

What do Positive Inequality Indices tell the Traveller?

	Income inequality in:	
	Anastasia	Zobeide
Variance	264	594
Coefficient of variation	0.214	0.214
Gini concentration index	**0.116**	**0.116**
Bonferroni index	0.151	0.151
UN Maximum equalisation percentage	0.084	0.084
Variance of logarithms	0.053	0.053
Mean logarithmic deviation	0.025	0.025
Theil index	0.024	0.024

Different values, but same inequality in Anastasia and Zobeide, as the Accountant had stated – except for the variance.

Measuring inequality historical notes: turning point no. 2

Dalton, 1920 (1)

- Two crucial insights in Dalton's (1920) seminal paper
  - Inequality can be defined as loss in social welfare generated by an unequal distribution of income (*utilitarian* social welfare)

  $$SW = \Sigma_i W(y_i)$$

  - 'Principle of transfers': inequality falls when a rank-preserving income transfer takes place from a richer to a poorer person
- Also discussion of some indices in terms of their properties (prelude to now dominant axiomatic approach).

Dalton, 1920 (2)

- Idea that inequality is loss in social welfare immediately challenged by Gini (1921):

  'the same methods are … applicable … to all other quantitative characteristics (economic, demographic, anatomical or physiological)'

- Main problem in application: inequality in utility space. As put by Yntema (1933), Dalton's procedure

  '… encounters the difficulty of finding the function which relates the individual's welfare to his income as well as the necessity of assuming identity between different individuals'

Measuring inequality historical notes: turning point no. 3

Atkinson, 1970 [& Kolm, 1969] (1)

- **First**: Criteria on social welfare functions to rank income distributions – parallel with 'stochastic dominance'
  - 1st order dominance: assume W increasing → compare cumulative distribution functions
  - 2nd order dominance: assume W increasing and concave → compare cumulative functions of cumulative distribution function → equivalent to compare Lorenz curves
  - Shorrocks (1983): compare generalised Lorenz curves
  - When ordering is ambiguous, we may use higher order dominance criteria with further restrictions on W, but still ordering is partial

Atkinson, 1970 [& Kolm, 1969] (2)

- **Second**: Recast Dalton's approach in income space by means of 'equally distributed equivalent income': level of income y_e which would give the same level of social welfare as the given distribution, when equally assigned to all individuals

  ▫ inequality index clears ambiguity of partial ordering

  ▫ inequality index $\rightarrow$ I=1–y_e/μ

  $$A=1-[\Sigma_i(y_i/\mu)^{(1-\varepsilon)}/n]^{1/(1-\varepsilon)} \qquad \text{for } \varepsilon\geq0, \varepsilon\neq1$$
  $$A=1-\Pi_i(y_i/\mu)^{(1/n)} \qquad \text{for } \varepsilon=1$$

  ▫ ε may represent concavity of utility function in a utilitarian interpretation, but also the social evaluator's judgement

  $\Rightarrow$ **ε** captures **aversion to inequality**: ε=0 implies A=0; higher ε implies greater aversion to inequality and A takes a higher value $\rightarrow$ A can embody different value judgements

What do Ethical Inequality Indices tell the Traveller?

	Income inequality in:	
	Anastasia	Zobeide
Indice di Atkinson, ε=0	0	0
Indice di Atkinson, ε=0,3	0.007	0.007
Indice di Atkinson, ε=1	0.025	0.025
Indice di Atkinson, ε=3	0.077	0.077

With ε=0, measured inequality is nil, but social welfare is SW=μ: as μ is 114 ducats in Zobeide and 76 ducats in Anastasia, better to be in Zobeide, as the **Shoemaker** had said.

But otherwise same inequality, as the **Accountant** had stated.

Yet, the variance is still an exception!

Measuring inequality historical notes: a little beaten path

The absolute dimension of inequality (1)

- Invariance to equal changes of income:
  - '**Scale independence**': indices unaffected by a proportionate increase (or decrease) of all incomes $\Rightarrow$ *all indices so far*
  - '**Translation independence**': indices unchanged by equal additions to (or subtractions from) all incomes
- Distinction made by Dalton, but little noticed.

The absolute dimension of inequality (2)

- Absolute criterion imaginatively advocated by Kolm (1976):

  'In May 1968 in France, radical students triggered a student upheaval which induced a workers' general strike. All this was ended by the Grenelle agreements which decreed a 13% increase in all payrolls. Thus, laborers earning 80 pounds a month received 10 pounds more, whereas executives who already earned 800 pounds a month received 100 pounds more. The Radicals felt bitter and cheated; in their view, this widely increased income inequality'

- But Atkinson (1983) cites sailors of the British Navy in 1931 who opposed a shilling a day reduction in their pay as

  '... they did not regard it as fair that they should bear a bigger proportionate cut than the officers'

The absolute dimension of inequality (3)

- Relative and absolute criteria a priori equally acceptable $\Rightarrow$ choice is again a value judgement

- People do differ in their views on inequality, and their views are far more complex than the simple relative/absolute dichotomy. See experiments by Amiel and Cowell (1999)

- 'Non-relative' measures:

  - Kolm (1976)

    $$K=\ln\{(1/n)\Sigma_i\exp[\kappa(\mu-y_i)]\}/\kappa \quad \text{for } \kappa>0 \qquad \text{('leftist' index)}$$
    $$C=\mu+\xi-[\Sigma_i(y_i+\xi)^{(1-\varepsilon)}/n]^{1/(1-\varepsilon)} \quad \text{for } \varepsilon>0,\ \varepsilon\neq1 \quad \text{('centrist' index)}$$
    $$C=\mu+\xi-\Pi_i(y_i+\xi)^{(1/n)} \quad \text{for } \varepsilon=1$$

  - Bossert and Pfingsten (1990):

    $$X=1+\xi-(1+\xi)\{\Sigma_i[(y_i+\xi)/(\mu+\xi)]^{(1-\varepsilon)}/n\}^{1/(1-\varepsilon)} \quad \text{for } \varepsilon>0,\ \varepsilon\neq1$$
 $$X=(1+\xi)[1-\Pi_i[(y_i+\xi)/(\mu+\xi)]^{(1/n)} \quad \text{for } \varepsilon=1$$

What do Absolute Ethical Inequality Indices tell the Traveller?

	Income inequality in:	
	Anastasia	Zobeide
Indice di Kolm, $\kappa=0.3$	20.6	33.6
Indice di Kolm, $\kappa=1$	24.4	37.4
Indice di Kolm, $\kappa=3$	25.5	38.5
Absolute Gini index (μG)	8.8	13.2

With an absolute index, inequality is higher in Zobeide than in Anastasia, as the **Physician** had concluded!

World poverty and inequality, 1820–1992
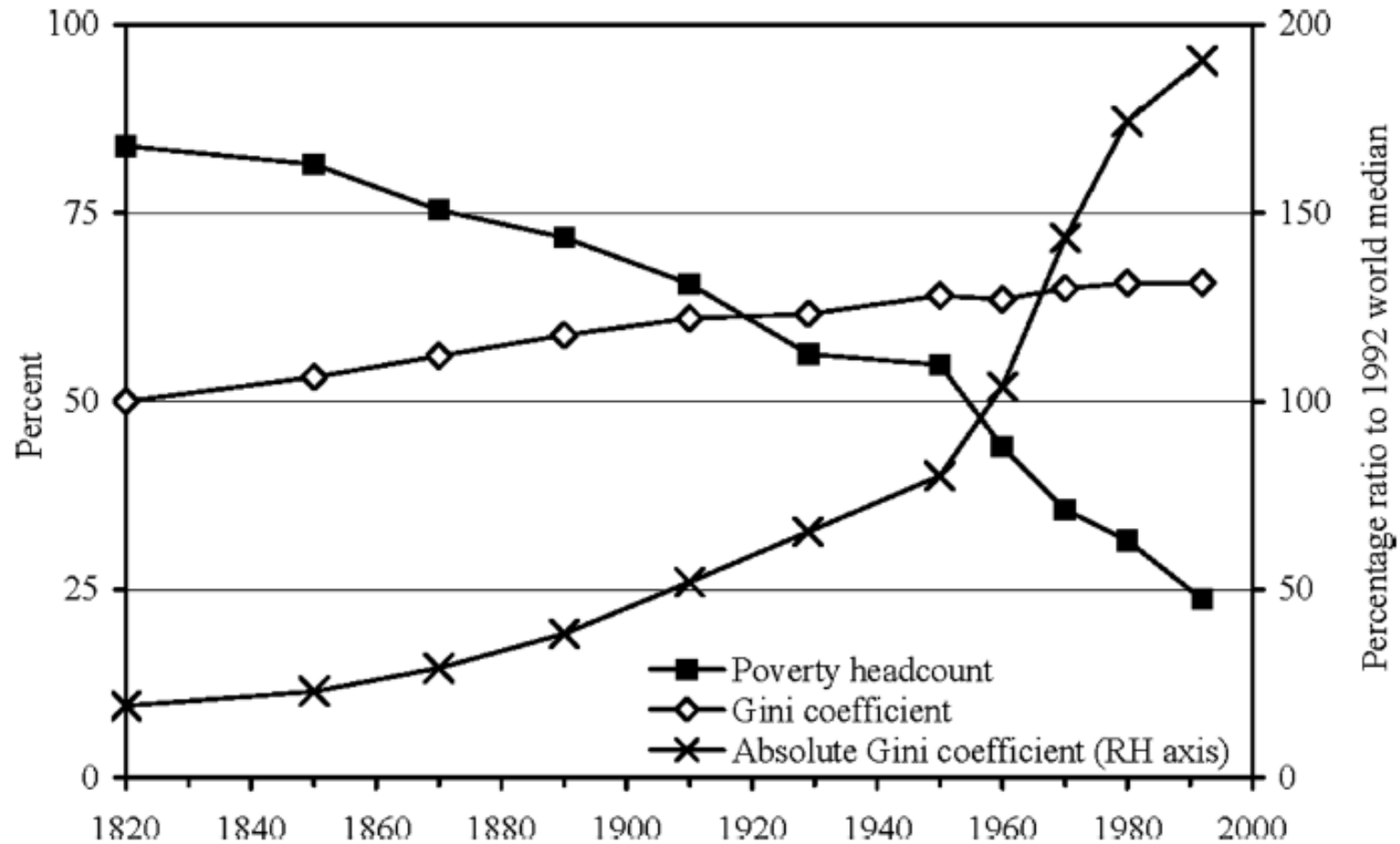
Source: Atkinson e Brandolini (2010).

Conclusions (1)

- Which index?
  - Indices differ for the weight they give to values located in different places on the income scale
  - Many indices are plausible and have informational content
  - Views about inequality are complex and include many aspects: a simple index, be it relative, absolute or intermediate, may be unable to embody all of them. Sometimes, we may want to rely on measures that allow for a more varied range of views (see Atkinson and Brandolini, 2010)
  - Apply dominance criteria, and be ready to accept partial orderings
  - Unlike Gini's view, choice of index may depend on variable under study $\rightarrow$ Relative or absolute index for life expectancy?

Conclusions (2)

- Gini's 'positive' approach to inequality measurement captures the objective diversity of incomes across persons that might be lost in some extreme formulations of ethical indices

- Dalton-Atkinson's 'normative' approach has helped to understand that a specific social judgement lies behind any inequality measure, implicit in the weight attributed to each single observation

- In all cases, bear in mind that normative and positive aspects are inextricably intertwined in inequality measurement

Thank you for your attention!