An ARIMA-Model-Based Approach to Seasonal Adjustment

S. C. HILLMER and G. C. TIAO*

This article proposes a model-based procedure to decompose a time series uniquely into mutually independent additive seasonal, trend, and irregular noise components. The series is assumed to follow the Gaussian ARIMA model. Properties of the procedure are discussed and an actual example is given.

KEY WORDS: ARIMA model; Seasonal adjustment; Census X-11 program; Pseudospectral density function; Model-based decomposition; Canonical decomposition.

1. INTRODUCTION

Business and economic time series frequently exhibit seasonality—periodic fluctuations that recur with about the same intensity each year. It has been argued (c.f., Nerlove, Grether, and Carvalho 1979, p. 147) that seasonality should be removed from economic time series so that underlying "business cycles" can be more easily studied and current economic conditions can be appraised. Of the large number of seasonal adjustment procedures, the most widely used is the Census X-11 method described in Shiskin, Young, and Musgrave (1967). The X-11 program and other methods that have been empirically developed tend to produce what their developers feel are desirable seasonal adjustments, but their statistical properties are difficult to assess from a theoretical viewpoint. Recently, there has been considerable interest in developing model-based procedures for the decomposition and seasonal adjustment of time series (see, e.g., the work of Grether and Nerlove 1970; Cleveland and Tiao 1976; Pierce 1978, 1980; Box, Hillmer, and Tiao 1978; Tiao and Hillmer 1978; and Burman 1980). Following this line of work and motivated in part by the considerations in the X-11 program, this article proposes a model-based approach that decomposes a time series into seasonal, trend, and irregular components.

We suppose that an observable time series at time t, Z_t, can be represented as

$$Z_t = S_t + T_t + N_t, \tag{1.1}$$

where S_t, T_t, and N_t are unobservable seasonal, trend, and noise components. It may be the case that a more accurate representation for Z_t would be as the product of S_t, T_t, and N_t. In that situation the model (1.1) would be appropriate for the logarithms of the original series. We assume that each of the components follows an ARIMA model,

$$\begin{aligned}\phi_S(B)S_t &= \eta_S(B)b_t\\ \phi_T(B)T_t &= \eta_T(B)c_t\\ \phi_N(B)N_t &= \eta_N(B)d_t,\end{aligned} \tag{1.2}$$

where B is the backshift operator such that $BS_t = S_{t-1}$, each of the pairs of polynomials $\{\phi_S(B), \eta_S(B)\}$, $\{\phi_T(B), \eta_T(B)\}$, and $\{\phi_N(B), \eta_N(B)\}$ have their zeros lying on or outside the unit circle and have no common zeros, and b_t, c_t, and d_t are three mutually independent white noise processes, identically and independently distributed as $N(0, \sigma_b^2)$, $N(0, \sigma_c^2)$, and $N(0, \sigma_d^2)$, respectively. Then it is readily shown that the overall model for Z_t is the ARIMA model

$$\varphi(B)Z_t = \theta(B)a_t, \tag{1.3}$$

where $\varphi(B)$ is the highest common factor of $\phi_S(B)$, $\phi_T(B)$, and $\phi_N(B)$, and $\theta(B)$ and σ_a^2 can be obtained from the relationship

$$\frac{\theta(B)\theta(F)\sigma_a^2}{\varphi(B)\varphi(F)} = \frac{\eta_S(B)\eta_S(F)\sigma_b^2}{\phi_S(B)\phi_S(F)} + \frac{\eta_T(B)\eta_T(F)\sigma_c^2}{\phi_T(B)\phi_T(F)} + \frac{\eta_N(B)\eta_N(F)\sigma_d^2}{\phi_N(B)\phi_N(F)}, \tag{1.4}$$

where $F = B^{-1}$. We also assume that the parameters in (1.3) are known. In practice a model for the observable series Z_t can be built from the data, and the estimated parameter values used as if they were the true values.

The ARIMA form has been found flexible enough to describe the behavior of many actual nonstationary and seasonal time series (Box and Jenkins 1970). There are situations in which such models by themselves may not be adequate; for example, a series describing employment may be dramatically affected by a strike and the model (1.3) does not cover such contingencies. However,

* S.C. Hillmer is Assistant Professor, School of Business, University of Kansas, Lawrence, KS 66044. G.C. Tiao is Professor of Statistics and Business, Department of Statistics, University of Wisconsin, Madison, WI 53706. This research was partially supported by the U.S. Bureau of the Census, Washington, D.C. under JSA No. 80-10; Army Research Office, Durham under grant No. DAAG29-78-G-0166; the ALCOA Foundation; and University of Kansas research allocation 3139-X0-0038. The article was completed when G.C. Tiao was on leave at the University of Chicago.

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in these situations ARIMA models can frequently be modified to approximate reality; for instance, intervention analysis techniques described in Box and Tiao (1975) might be used to account for the effects of strikes and other exogenous events.

Given the observable Z_t and the structure in (1.1), (1.2), and (1.3), the problem is to decompose Z_t into S_t, T_t, and N_t. Our approach is as follows: (a) We first impose restrictions on $\phi_S(B)$ and $\phi_T(B)$ for the component models (1.2) based in part on considerations in the Census X-11 program. (b) A model for Z_t is derived from observable data. (c) A principle is adopted that uniquely specifies the component models in a manner consistent with the imposed restrictions and the model derived for Z_t. (d) Given the component models, known signal extraction methods are applied to decompose Z_t into (estimates of) the components. Properties of the procedure are explored and an illustration using an actual time series is presented.

2. DECOMPOSITION WHEN THE COMPONENT MODELS ARE KNOWN

If in (1.1) the stochastic structures of S_t, T_t, and N_t in (1.2) are known, then estimates of S_t and T_t can be readily obtained (see, e.g., Whittle 1963 and Cleveland and Tiao 1976). Specifically, Cleveland and Tiao have shown that, when all the zeros of $\phi_S(B)$, $\phi_T(B)$, and $\phi_N(B)$ are on or outside of the unit circle, the minimum mean squared estimates of the seasonal and trend components S_t and T_t are, respectively,

$$\hat{S}_t = W_S(B)Z_t \quad \text{and} \quad \hat{T}_t = W_T(B)Z_t, \tag{2.1}$$

where

$$W_S(B) = \frac{\sigma_b^2}{\sigma_a^2}\frac{\varphi(B)\varphi(F)\eta_S(B)\eta_S(F)}{\theta(B)\theta(F)\phi_S(B)\phi_S(F)}$$

and

$$W_T(B) = \frac{\sigma_c^2}{\sigma_a^2}\frac{\varphi(B)\varphi(F)\eta_T(B)\eta_T(F)}{\theta(B)\theta(F)\phi_T(B)\phi_T(F)}\,.$$

Because in practice the S_t, T_t, and N_t series are unobservable, it is usually unrealistic to assume that the component models in (1.2) are known. As a result, the weight functions $W_S(B)$ and $W_T(B)$ cannot be determined and the values $\hat{S}_t$ and $\hat{T}_t$ cannot be calculated. We can, however, get an accurate estimate of the model (1.3) from the observable Z_t series. Consequently, it is of interest to investigate to what extent a known model for Z_t will determine the models for the component series.

3. PROPERTIES OF SEASONAL AND TREND COMPONENTS

It is well known that the Census X-11 procedure may be approximated by a linear filter (for instance see Young 1968 and Wallis 1974). One important feature of the X-11 filter weights for the trend and the seasonal components is that the weights applied to observations more removed from the current time period decrease. This feature was incorporated into the X-11 program probably because of the belief that the trend and seasonal components of many series change over time; consequently the information about the current trend or seasonal is contained in the values of Z_t close to current time. Therefore, in developing a decomposition procedure we should allow for evolving trend and seasonal components.

Stochastic Trend

Economic data often exhibit underlying movements that drift over time. While *locally* such movements might be adequately modeled by a polynomial in time, a fixed polynomial time function is clearly inappropriate over the *entire* time span. Thus a stochastic trend model is needed, and we assume that the trend component, T_t, follows the nonstationary model

$$(1 - B)^d T_t = \eta_T(B)c_t, \tag{3.1}$$

where $\eta_T(B)$ is a polynomial in B of degree at most d, and c_t are iid $N(0, \sigma_c^2)$. Box and Jenkins (1970, p. 149) have shown that the minimum mean squared error forecast function of (3.1) is a polynomial time function of degree $(d - 1)$ whose coefficients are updated as the origin of forecast is advanced; therefore (3.1) can be regarded as a polynomial model with stochastic coefficients.

It is also of interest to consider the trend component in the frequency domain. Intuitively, the spectral density function of a trend component should be large for the low frequencies and small for higher frequencies. Since the model (3.1) is nonstationary, the spectral density function is strictly speaking not defined. However, we can define a pseudospectral density function (psdf) for (3.1) by

$$f_T(w) = \sigma_c^2\eta_T(e^{iw})\eta_T(e^{-iw})/(1 - e^{iw})^d(1 - e^{-iw})^d,$$
$$0 \le w \le \pi. \tag{3.2}$$

Now the psdf (3.2) is infinite at $w = 0$ and very large for small w. This is consistent with what could be viewed as a stochastic trend component.

Stochastic Seasonal

A deterministic seasonal component S_t of period s would have the property that it repeats itself every s periods and that the sum of any s consecutive components should be a constant, that is,

$$S_t = S_{t-s} \quad \text{and} \quad U(B)S_t = c, \tag{3.3}$$

where $U(B) = 1 + B + \ldots + B^{s-1}$ and c is an arbitrary constant that can be taken as zero. Such a model, however, implies that the seasonal pattern is fixed over time. For business and economic time series, it seems reasonable to require that the seasonal component should be capable of evolving over time but that *locally* a regular seasonal pattern should be preserved. In other words, $U(B)S_t$ should be random but cluster about zero. Consider the nonstationary model

$$U(B)S_t = \eta_S(B)b_t, \tag{3.4}$$

where $\eta_S(B)$ is a polynomial in B of degree at most $s - 1$ and b_t are iid $N(0, \sigma_b^2)$. That is, the consecutive moving sum of s components, $U(B)S_t$, follows a moving average model of order (at most) $s - 1$. It is readily shown that the forecasting function of (3.4) at a given time origin follows a fixed seasonal pattern of period s, but the pattern is updated as the origin is advanced. Also, $EU(B)S_t = E\eta_S(B)b_t = 0$. Thus, the model (3.4) preserves a local cyclical pattern but allows seasonality to evolve over time.

It is also informative to consider the psdf, $f_S(w)$, of the model in (3.4)

$$f_S(w) = \sigma_b^2 \frac{\eta_S(e^{iw})\eta_S(e^{-iw})}{U(e^{iw})U(e^{-iw})}. \tag{3.5}$$

It can be shown that $f_S(w)$ has the following properties: (a) $f_S(w)$ is infinite at the seasonal frequencies $w = 2k\pi/s$ for $k = 1, \ldots, [s/2]$, where $[x]$ denotes the greatest integer less than or equal to x; (b) $f_S(w)$ has relative minimum at $w = 0$ and near the frequencies $w = ((2k - 1)\pi)/s$ for $k = 2, \ldots, [s/2]$. Therefore, the psdf of (3.4) has infinite power at the seasonal frequencies and relatively small power away from the seasonal frequencies.

4. MODEL-BASED SEASONAL DECOMPOSITION

From considerations in the previous section, we require $\varphi(B)$ to contain the factor $U(B)$ before we impose a seasonal component S_t and to contain the factor $(1 - B)^d$ before we impose a trend component T_t for Z_t. We further require that in (1.2) the autoregressive polynomial of N_t, $\phi_N(B)$, has no common zeros with either $(1 - B)^d$ or $U(B)$, because otherwise it would imply the existence of additional seasonal and trend components that could then be absorbed into S_t and T_t. Thus, we shall suppose that in (1.3)

$$\varphi(B) = (1 - B)^d U(B)\phi_N(B), \tag{4.1}$$

where the three factors on the right side have no common zeros. In other words, knowing the model for Z_t and assuming that a decomposition is possible, the autoregressive polynomials of S_t, T_t, and N_t can be uniquely determined. Also, the relationship (1.4) becomes

$$\frac{\theta(B)\theta(F)}{\varphi(B)\varphi(F)}\sigma_a^2 = \frac{\eta_S(B)\eta_S(F)}{U(B)U(F)}\sigma_b^2 + \frac{\eta_T(B)\eta_T(F)}{(1 - B)^d(1 - F)^d}\sigma_c^2 + \frac{\eta_N(B)\eta_N(F)}{\phi_N(B)\phi_N(F)}\sigma_d^2. \tag{4.2}$$

The more difficult task is to determine the moving average polynomials and the innovation variances. Within the class of $\eta_S(B)$ and $\eta_T(B)$ whose degrees are at most $(s - 1)$ and d as required by (3.1) and (3.4), any choice of the three moving average polynomials $\eta_S(B)$, $\eta_T(B)$, and $\eta_N(B)$ and the three variances σ_b^2, σ_c^2, and σ_d^2 satisfying (4.2) will be called an *acceptable* decomposition because it is consistent with information provided by the model for the observed data Z_t.

We now give a necessary and sufficient condition for the existence of an acceptable decomposition. Assuming that $\varphi(B)$ takes the form (4.1), we may perform a unique partial fraction decomposition of the left side of (4.2) to yield

$$\frac{\theta(B)\theta(F)}{\varphi(B)\varphi(F)}\sigma_a^2 = \frac{Q_S(B)}{U(B)U(F)} + \frac{Q_T(B)}{(1 - B)^d(1 - F)^d} + \frac{Q_N(B)}{\phi_N(B)\phi_N(F)}, \tag{4.3}$$

where

$$Q_S(B) = q_{0S} + \sum_{i=1}^{s-2} q_{iS}(B^i + F^i),$$

$$Q_T(B) = q_{0T} + \sum_{i=1}^{d-1} q_{iT}(B^i + F^i),$$

and $Q_N(B)$ can be obtained by subtraction. The uniqueness in (4.3) results from the fact that the degrees of $Q_S(B)$ and $Q_T(B)$ are lower than the degrees of the corresponding denominator. Now for $0 \le w \le \pi$, let

$$\epsilon_1 = \min_w \frac{Q_S(e^{-iw})}{|U(e^{-iw})|^2},$$

$$\epsilon_2 = \min_w \frac{Q_T(e^{-iw})}{|1 - e^{-iw}|^{2d}}, \tag{4.4}$$

and

$$\epsilon_3 = \min_w \frac{Q_N(e^{-iw})}{|\phi_N(e^{-iw})|^2}.$$

We now show that an acceptable decomposition exists if and only if $\epsilon_1 + \epsilon_2 + \epsilon_3 \ge 0$.

Proof. By writing $B = e^{-iw}$, $0 \le w \le \pi$, each of the three terms on the right side of (4.2) is a psdf.

Since $\eta_S(B)$ is of degree at most $s - 1$ and $\eta_T(B)$ is of degree at most d, by comparing (4.2) with (4.3) we can write

$$\frac{|\eta_S(e^{-iw})|^2\sigma_b^2}{|U(e^{-iw})|^2} = \frac{Q_S(e^{-iw})}{|U(e^{-iw})|^2} + \gamma_1,$$

$$\frac{|\eta_T(e^{-iw})|^2\sigma_c^2}{|1 - e^{-iw}|^{2d}} = \frac{Q_T(e^{-iw})}{|1 - e^{-iw}|^{2d}} + \gamma_2, \tag{4.5}$$

and

$$\frac{|\eta_N(e^{-iw})|^2\sigma_d^2}{|\phi_N(e^{-iw})|^2} = \frac{Q_N(e^{-iw})}{|\phi_N(e^{-iw})|^2} + \gamma_3,$$

where γ_1, γ_2, and γ_3 are three constants such that $\gamma_1 + \gamma_2 + \gamma_3 = 0$. The constants γ_i provide a means to change from the initial partial fractions decomposition (4.3) to an acceptable decomposition if one exists. Thus, an acceptable decomposition implies and is implied by the fact that $\gamma_i + \epsilon_i \geq 0$ for $i = 1, 2, 3$ or equivalently that $\epsilon_1 + \epsilon_2 + \epsilon_3 \geq 0$.

From the previous discussion, when $\epsilon_1 + \epsilon_2 + \epsilon_3 \geq 0$, every set of γ_i's corresponds to a unique acceptable decomposition; thus a unique decomposition exists if and only if $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$. On the other hand, when $\epsilon_1 + \epsilon_2 + \epsilon_3 > 0$, there are an infinite number of ways of adding constants to the three terms on the right side of (4.3) to obtain acceptable decompositions.

5. A CANONICAL DECOMPOSITION

In the absence of prior knowledge about the precise stochastic structure of the trend and seasonal components, all of the information in the known model of Z_t, (1.3), about S_t and T_t is embodied in (4.2). However, when $\epsilon_1 + \epsilon_2 + \epsilon_3 > 0$, this information is not sufficient to uniquely determine the models for S_t and T_t. To perform seasonal adjustment of the data, an arbitrary choice must be made. Considering that the seasonal and trend components should be slowly evolving, it seems reasonable to extract as much white noise as possible from the seasonal and trend components subject to the restrictions in (4.2). Thus, we seek to maximize the innovation variance σ_d^2 of the noise component N_t. Therefore, we define the *canonical decomposition* as the decomposition that maximizes σ_d^2 subject to the restrictions in (4.2).

Properties of the Canonical Decomposition

In the following we denote the canonical seasonal component by $\tilde{S}_t$, the canonical trend component by $\tilde{T}_t$, and use the same convention when referring to the moving average polynomials and innovation variances of the canonical decompositions. We prove the following properties of the canonical decomposition in the appendix. (a) The canonical decomposition is unique. (b) It minimizes the innovation variances σ_b^2 and σ_c^2. (c) The polynomials $\tilde{\eta}_S(B)$ and $\tilde{\eta}_T(B)$ have at least one zero on the unit circle so that the models for $\tilde{S}_t$ and $\tilde{T}_t$ are noninvertible. (d) If $\hat{S}_t$ and $\hat{T}_t$ are any acceptable seasonal and trend components other than the canonical decomposition, then $\hat{S}_t = \tilde{S}_t + e_t$ and $\hat{T}_t = \tilde{T}_t + \alpha_t$, where e_t and α_t are white noise series. (e) The variance of $U(B)S_t$ is minimized for the canonical decomposition.

One may lend justification of the (arbitrary) choice of the canonical decomposition on the basis of these properties. In particular, property (b) is intuitively pleasing since the randomness in S_t arises from the sequence of b_t's and the randomness in T_t arises from the sequence of c_t's. Thus, minimizing σ_b^2 and σ_c^2 makes the seasonal and trend components as deterministic as possible while remaining consistent with the information in the observable Z_t series. Also, from property (d) any acceptable seasonal component can be viewed as the sum of the canonical seasonal and white noise. But $\tilde{S}_t$ is a highly predictable component that accounts for all of the seasonality in the original series and e_t is a completely unpredictable component. Thus, one might argue that the choice of an acceptable decomposition other than the canonical decomposition only produces a more confused seasonal component than necessary. Finally, property (e) is intuitively pleasing since $E[U(B)S_t] = 0$ and a small value for $\text{var}[U(B)S_t]$ will help ensure that the sum of s consecutive seasonal components remains close to zero.

6. APPLICATION TO SOME SPECIAL SEASONAL MODELS

We now illustrate the results in the preceding sections with the following three special cases of (1.3). These models have been frequently used in practice to fit seasonal data (see, e.g., Box and Jenkins 1970, and Tiao, Box, and Hamming 1975).

$$(1 - B^s)Z_t = (1 - \theta_2 B^s)a_t, \tag{6.1}$$

$$(1 - B)(1 - B^s)Z_t = (1 - \theta_1 B)(1 - \theta_2 B^s)a_t, \tag{6.2}$$

and

$$(1 - B^s)Z_t = (1 - \theta_1 B)(1 - \theta_2 B^s)a_t. \tag{6.3}$$

Without loss of generality, we assume that $\sigma_a^2 = 1$. For these models, the general approach is as follows. We first divide the denominator of the left side of (4.3) into the numerator to obtain $Q_N(B)$ and a remainder term $R(B)$; we then perform a partial fractions expansion of $R(B)/\varphi(B)\varphi(F)$ to obtain $Q_S(B)$ and $Q_T(B)$; and finally we find the minimum values ϵ_1, ϵ_2, and ϵ_3 in order to investigate whether an acceptable decomposition exists.

The model (6.1)

In this case, $d = 1$ and $\phi_N(B) = 1$. By partial fraction, (4.3) becomes

$$\frac{(1 - \theta_2 B^s)(1 - \theta_2 F^s)}{(1 - B^s)(1 - F^s)} = \frac{Q_S(B)}{U(B)U(F)} + \frac{Q_T(B)}{(1 - B)(1 - F)} + \theta_2, \tag{6.4}$$

where

$$Q_T(B) = \frac{1}{s^2}(1 - \theta_2)^2$$

and

$$Q_S(B) = (1 - \theta_2)^2 \left[1 - \frac{1}{s^2} U(B)U(F)\right] \Big/ (1 - B)(1 - F)$$

$$= \frac{1}{6s^2}(1 - \theta_2)^2 \times \left[\sum_{l=2}^{s-1} (l - 1)l(l + 1)(B^{s-l} + F^{s-l}) + (s - 1)s(s + 1)\right].$$

For the trend component, we see that $Q_T(e^{-iw}) \mid 1 - e^{-iw} \mid^{-2}$ is monotonically decreasing in w and

$$\epsilon_2 = \frac{1}{4s^2}(1 - \theta_2)^2.$$

For the seasonal component, it is easy to show that $Q_S(e^{-iw}) \mid U(e^{-iw}) \mid^{-2} \geq 0$ and has a local minimum at $w = 0$. Also, we conjecture that $w = 0$ is in fact the global minimum. This conjecture is verified analytically for $s \leq 3$ and numerically for s from 4 to 20. Assuming this is true for all s we find that

$$\min_w \left\{1 - \frac{1}{s^2} \mid U(e^{-iw}) \mid^2\right\} \Big/ \mid 1 - e^{-iw} \mid^2 = (s^2 - 1)/12 \qquad (6.5)$$

so that $\epsilon_1 = (1 - \theta_2)^2(s^2 - 1)/12s^2$. Since $\epsilon_3 = \theta_2$, for an acceptable decomposition to exist it is required that

$$\epsilon_1 + \epsilon_2 + \epsilon_3 = \theta_2 + \frac{(1 - \theta_2)^2}{4s^2} + \frac{(s^2 - 1)(1 - \theta_2)^2}{12s^2} \geq 0$$

or equivalently

$$\theta_2 \geq -\frac{(5s^2 - 2) + 2s\sqrt{6(s^2 - 1)}}{(s^2 + 2)}. \qquad (6.6)$$

Values of the lower bound of θ_2 for selected values of s are given in the following tabulation:

s	2	4	6
l.b.θ_2	−.1716	−.1170	−.1080
8	10	12	∞
−.1049	−.1035	−.1027	−.1010

Therefore, there are values of θ_2 for which the model (6.1) is not consistent with an additive decomposition as we have defined it; however, a value of $\theta_2 > -.1010$ will always lead to an acceptable decomposition.

When strict inequality is obtained in (6.6), there will be an infinite number of acceptable decompositions. The canonical decomposition corresponds to

$$\frac{\bar\sigma_b^2 \bar\eta_S(B)\bar\eta_S(F)}{U(B)U(F)} = \frac{Q_S(B)}{U(B)U(F)} - \frac{s^2 - 1}{12s^2}(1 - \theta_2)^2$$

and (6.7)

$$\frac{\bar\sigma_c^2 \bar\eta_T(B)\bar\eta_T(F)}{(1 - B)(1 - F)} = \frac{1}{4s^2}(1 - \theta_2)^2 \frac{(1 + B)(1 + F)}{(1 - B)(1 - F)}.$$

The Model (6.2)

For this model, $d = 2$ and $\phi_N(B) = 1$. After some algebraic reduction, we find

$$\frac{(1 - \theta_1 B)(1 - \theta_2 B^s)(1 - \theta_1 F)(1 - \theta_2 F^s)}{(1 - B)(1 - B^s)(1 - F)(1 - F^s)} = \frac{Q_S(B)}{U(B)U(F)} + \frac{Q_T(B)}{(1 - B)^2(1 - F)^2} + \theta_1\theta_2, \qquad (6.8)$$

where

$$Q_T(B) = \frac{(1 - \theta_1)^2(1 - \theta_2)^2}{s^2} \times \left\{1 + \left[\frac{\theta_2 s^2}{(1 - \theta_2)^2} + \frac{(s^2 - 4)}{12} + \frac{(1 + \theta_1)^2}{4(1 - \theta_1)^2}\right] \times (1 - B)(1 - F)\right\}$$

and

$$(1 - \theta_2)^{-2}(1 - B)^2(1 - F)^2 Q_S(B) = (1 - \theta_1)^2\left\{1 - \frac{1}{s^2}U(B)U(F)\right\} + \theta_1(1 - B)(1 - F) - \left\{\frac{s^2 - 4}{12s^2}(1 - \theta_1)^2 + \frac{(1 + \theta_1)^2}{4s^2}\right\}(1 - B^s)(1 - F^s).$$

We now show that an acceptable decomposition exists if $\theta_2 \geq 0$.

Proof. First, setting $B = -1$ (or $w = \pi$ in $B = e^{-iw}$) in $Q_T(B)(1 - B)^{-2}(1 - F)^{-2}$, we have

$$\frac{Q_T(-1)}{16} = \frac{(1 - \theta_2)^2}{48s^2} \times \{(1 - \theta_1)^2(s^2 - 1) + 3(1 + \theta_1)^2\} + \frac{\theta_2(1 - \theta_1)^2}{4} = C \qquad (6.9)$$

say. The right side of (6.8) can now be written as

$$\frac{Q_S{}^*(B)}{U(B)U(F)} + \frac{Q_T{}^*(B)}{(1 - B)^2(1 - F)^2} + \theta_2\frac{(1 + \theta_1)^2}{4}, \qquad (6.10)$$

where

$$Q_T{}^*(B) = Q_T(B) - C(1 - B)^2(1 - F)^2$$

and

$$Q_S{}^* = Q_S(B) + U(B)U(F)\left\{C - \frac{\theta_2(1 - \theta_1)^2}{4}\right\}.$$

Also, it can be verified that

$$(1 - \theta_2)^{-2}(1 - B)^2(1 - F)^2 Q_S{}^*(B) = \frac{(1 - \theta_1)^2}{4}(1 + B)(1 + F) \times \left\{1 - \frac{1}{s^2}U(B)U(F) - \frac{s^2 - 1}{12s^2}(1 - B^s)(1 - F^s)\right\} + \frac{(1 + \theta_1)^2}{4}(1 - B)(1 - F) \times \left\{1 - \frac{1}{4s^2}U(B)U(F)(1 + B)(1 + F)\right\}. \qquad (6.11)$$

When $\theta_2 \geq 0$, one can readily show that $Q_T(e^{-iw}) \mid 1 - e^{-iw} \mid^{-2}$ is monotonically decreasing in w so that the second term in (6.10) is nonnegative for all w. Now, on the right side of the equation in (6.11), the second term with $B = e^{-iw}$ is clearly nonnegative for all w and, from (6.5), so is the first term. Thus, an acceptable decomposition exists and is given by (6.10).

The Model (6.3)

In this case, $d = 1$ and $\phi_N(B) = 1$. By partial fraction, we find

$$\frac{(1-\theta_1 B)(1-\theta_1 F)(1-\theta_2 B^s)(1-\theta_2 F^s)}{(1-B^s)(1-F^s)} = \frac{Q_S(B)}{U(B)U(F)} + \frac{Q_T(B)}{(1-B)(1-F)} + Q_N(B), \quad (6.12)$$

where

$$Q_T(B) = \frac{1}{s^2}(1-\theta_1)^2(1-\theta_2)^2,$$

$$(1-\theta_2)^{-2}(1-B)(1-F)Q_S(B) = \frac{(1+\theta_1)^2}{4}(1-B)(1-F) + (1-\theta_1)^2\left\{\frac{1}{4}(1+B)(1+F) - \frac{1}{s^2}U(B)U(F)\right\},$$

and

$$Q_N(B) = \theta_2(1-\theta_1 B)(1-\theta_1 F).$$

Noting that

$$\min_w Q_T(e^{-iw})\,|1-e^{-iw}|^{-2} = \frac{1}{4s^2}(1-\theta_1)^2(1-\theta_2)^2,$$

we can express the right side of (6.12) alternatively as

$$\frac{Q_S^*}{U(B)U(F)} + \frac{Q_T^*(B)}{(1-B)(1-F)} + Q_N^*(B), \quad (6.13)$$

where

$$Q_T^*(B) = Q_T(B) - \frac{1}{4s^2}(1-\theta_1)^2 \times (1-\theta_2)^2(1-B)(1-F),$$

$$Q_N^*(B) = Q_N(B) + \frac{1}{4s^2}(1+\theta_1)^2(1-\theta_2)^2,$$

and

$$(1-\theta_2)^{-2}(1-B)(1-F)Q_S^*(B) = (1-\theta_1 B)(1-\theta_1 F)\left\{1 - \frac{1}{s^2}U(B)U(F)\right\}.$$

Similar to the model (6.2), when $\theta_2 \geq 0$, all three terms in (6.13) are nonnegative for all w so that acceptable decompositions exist.

For the models (6.2) and (6.3), acceptable decompositions also exist for negative values of θ_2 near zero. The precise lower bounds are difficult to determine analytically. However, for these as well as for any model of the form (1.3) satisfying the condition (4.1), the existence of acceptable decompositions and the corresponding canonical form can always be determined by numerical methods. A computer program to determine the canonical component models and to compute the estimates $\hat{S}_t$, $\hat{T}_t$, and $\hat{N}_t$ is available on request.

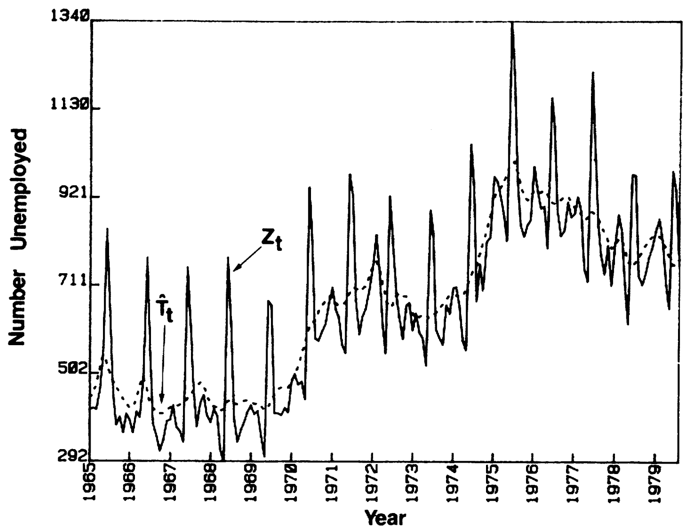

Figure 1. Monthly Unemployed Males Aged 16 to 19 (January 1971–August 1979) and the Estimated Trend Component Series

7. AN EXAMPLE

We now apply the model-based decomposition procedure to the monthly series of U.S. unemployed males aged 16 to 19 from January 1965 to August 1979, obtained from the Bureau of Labor Statistics. The series is a component used in constructing the monthly unemployment index.

The series is plotted in Figure 1. The variability of the series appears relatively constant over time; thus we decided to model the series in the original metric. It is found that the data can be adequately represented by the model (6.2) with

$$s = 12, \quad \hat{\theta}_1 = \underset{(.075)}{.313}, \quad \text{and} \quad \hat{\theta}_2 = \underset{(.035)}{.817}, \quad (7.1)$$

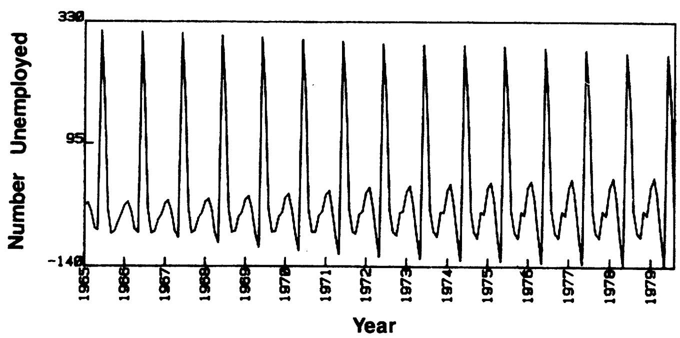

Figure 2. Estimated Seasonal Component Series for the Unemployed Males Data

Table 1. Weight Function for Estimating the Seasonal Component: Unemployed Males Data

Lag j	w_j											
0–11	.085	−.007	−.008	−.008	−.008	−.008	−.008	−.007	−.007	−.007	−.007	−.007
12–23	.076	−.007	−.007	−.007	−.006	−.006	−.006	−.006	−.006	−.006	−.006	−.006
24–35	.062	−.006	−.005	−.005	−.005	−.005	−.005	−.005	−.005	−.005	−.005	−.005
36–47	.051	−.005	−.004	−.004	−.004	−.004	−.004	−.004	−.004	−.004	−.004	−.004

with the standard errors of the parameter estimates given in parentheses below the estimates.

Assuming the estimates in (7.1) are the true values, we computed the corresponding canonical decomposition and, from (2.3), the associated weights for the estimates of the seasonal and trend components. These weights are given in Tables 1 and 2 from the center through lag 47. In both cases the remaining weights can be obtained by using the equation $w_j = .313\ w_{j-1} + .817\ w_{j-12} - .256\ w_{j-13}$. We observe that the weights associated with the seasonal component die out slowly and span a large number of years. This is in contrast to the weights associated with the standard Census X-11 program whose weights die out in about three years (see, e.g., Wallis 1974). We note that the rate at which the weight in the model-based approach decreases is primarily determined by the value of the parameter $\hat{\theta}_2 = .817$, which is determined from the original series.

The estimated trend component $\hat{T}_t$ is shown in Figure 1 and the estimated seasonal component $\hat{S}_t$ is plotted in Figure 2. We make the following observations. (a) The estimated trend component appears to capture the basic underlying movements of the series. (b) The seasonal component seems to have been adequately removed by the model-based decomposition. (c) The estimated seasonal component varies around a zero level and it is slowly changing over time. Therefore, for this particular series it appears that the model-based seasonal adjustment procedure has led to intuitively pleasing results.

8. DISCUSSION

In this article, we have proposed a model-based procedure to decompose a time series uniquely into mutually independent seasonal, trend, and irregular noise components. The method can be readily extended to models other than the ones discussed. For example, when $s = 12$, the autoregressive part of the seasonal component need not be $U(B)$, but can be any product of the factors $(1 + B)$, $(1 + B^2)$, $(1 + B + B^2)$, $(1 - B + B^2)$, $(1 + \sqrt{3}B + B^2)$, and $(1 - \sqrt{3}B + B^2)$. Also, the trend component may be augmented into a "trend-cycle" component by allowing the autoregressive part to take the form $(1 - B)^d \phi_T^*(B)$, where $\phi_T^*(B)$ has all its zeros lying on the unit circle (but distinct from $B = 1$ and those of the seasonal component). The possibilities are unlimited, depending on the form of the known model of Z_t and the nature of the problem.

Finally, we remark here that in illustrating the decomposition procedure with the models (6.1) to (6.3), in each case the values of θ_2 are restricted essentially to be nonnegative to yield acceptable decompositions. While we have rarely seen in practice a negative estimate of θ_2, it is conceivable that this could happen. One possible explanation for a negative θ_2 is that the white noise b_t and c_t for the seasonal and trend components are correlated. As an extreme example of the model (6.1) with $s = 2$, suppose the component models are

$$(1 + B)S_t = (1 - B)b_t,$$
$$(1 - B)T_t = (1 + B)c_t, \tag{8.1}$$

and

$$N_t \equiv 0.$$

The reader can readily verify that if $\sigma_b^2 = \sigma_c^2$ and b_t and c_t are perfectly positively correlated, then $\theta_2 = -1$. Thus, by allowing the component models to be dependent, we could increase the range of the models of Z_t for which acceptable decompositions exist. This seems to be an interesting topic for further study.

APPENDIX

In this appendix we sketch the proof of the properties of the canonical decomposition given in Section 5. Upon multiplying each expression in (4.5) by the denominator on the left side of the corresponding equation, we obtain

Table 2. Weight Function for Estimating the Trend Component: Unemployed Males Data

j	w_j											
0–11	.318	.212	.072	.028	.014	.010	.008	.008	.007	.005	.001	−.012
12–23	−.021	−.012	.001	.005	.006	.006	.006	.006	.006	.004	.001	−.009
24–35	−.018	−.010	.001	.004	.005	.005	.005	.005	.005	.004	.001	−.008
36–47	−.014	−.008	.001	.003	.004	.004	.004	.004	.004	.003	.001	−.006

$$
\begin{aligned}
|\eta_S(e^{-iw})|^2 \sigma_b^2 &= Q_S(e^{-iw}) + \gamma_1 |U(e^{-iw})|^2 \\
&= f_S(w, \gamma_1), \\
|\eta_T(e^{-iw})|^2 \sigma_c^2 &= Q_T(e^{-iw}) + \gamma_2 |1 - e^{-iw}|^{2d} \qquad \text{(A.1)} \\
&= f_T(w, \gamma_2), \\
|\eta_N(e^{-iw})|^2 \sigma_d^2 &= Q_N(e^{-iw}) + \gamma_3 |\phi_N(e^{-iw})|^2 \\
&= f_N(w, \gamma_3).
\end{aligned}
$$

Using a result of Hannan (1970, p. 137), we have that

$$
\begin{aligned}
\sigma_b^2(\gamma_1) &= \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln f_S(w, \gamma_1)\, dw\right\}, \\
\sigma_c^2(\gamma_2) &= \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln f_T(w, \gamma_2)\, dw\right\}, \qquad \text{(A.2)} \\
\sigma_d^2(\gamma_3) &= \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln f_N(w, \gamma_3)\, dw\right\}.
\end{aligned}
$$

Now in (A.1), $f_N(w, \gamma_3)$ does not depend on γ_3 if $\phi_N(e^{-iw}) = 0$ and is otherwise strictly increasing in γ_3; thus σ_d^2 is maximized when $\gamma_3 = \epsilon_1 + \epsilon_2$. From the restrictions that $\gamma_1 + \gamma_2 + \gamma_3 = 0$ and $\gamma_i + \epsilon_i \geq 0$, $i = 1, 2, 3$, we have that for the canonical decomposition $\gamma_1 = -\epsilon_1$ and $\gamma_2 = -\epsilon_2$. Therefore, the canonical decomposition is unique and furthermore, from (A.1) and (A.2), the innovation variances $\sigma_b^2(\gamma_1)$ and $\sigma_c^2(\gamma_2)$ are minimized for the canonical decomposition. In addition, if we take $\gamma_1 = -\epsilon_1$ and $\gamma_2 = -\epsilon_2$ in (A.1), both $f_S(w, -\epsilon_1)$ and $f_S(w, -\epsilon_2)$ are zero for some $0 \leq w \leq \pi$ implying that $\bar{\eta}_S(B)$ and $\bar{\eta}_T(B)$ are not invertible.

If we let

$$f_{\bar{S}}(w) = Q_S(e^{-iw}) \,|U(e^{-iw})|^{-2} - \epsilon_1$$

denote the psdf of $\bar{S}_t$ and let $f_{\tilde{S}}(w)$ denote the psdf of any other acceptable decomposition $\tilde{S}_t$, then it follows that

$$f_{\tilde{S}}(w) = f_{\bar{S}}(w) + \sigma_e^2 \qquad \text{(A.3)}$$

with $\sigma_e^2 > 0$. Equation (A.3) implies $\tilde{S}_t = \bar{S}_t + e_t$, where e_t is white noise with variance σ_e^2.

Finally, from (4.5) the variance of $U(B)S_t$ is

$$\operatorname{var}[U(B)S_t] = \frac{1}{2\pi}\int_{-\pi}^{\pi} [Q_S(e^{-iw}) + \gamma_1 |U(e^{-iw})|^2]\, dw. \qquad \text{(A.4)}$$

It is evident that (A.4) is minimized when γ_1 is made as small as possible or $\gamma_1 = -\epsilon_1$, the value corresponding to the canonical decomposition.

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